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Curvature on determinant bundles and first Chern forms

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Abstract

The Quillen–Bismut–Freed construction associates a determinant line bundle with connection to an infinite dimensional super vector bundle with a family of Dirac-type operators. We define the regularized first Chern form of the infinite dimensional bundle, and relate it to the curvature of the Bismut–Freed connection on the determinant bundle. In finite dimensions, these forms agree (up to sign), but in infinite dimensions there is a correction term, which we express in terms of Wodzicki residues.

We illustrate these results with a string theory computation. There is a natural super vector bundle over the manifold of smooth almost complex structures on a Riemannian surface. The Bismut–Freed superconnection is identified with classical Teichmüller theory connections, and its curvature and regularized first Chern form are computed.

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1. Introduction

A finite rank Hermitian super vector bundle $\mathcal{E} = \mathcal{E}^+ \oplus \mathcal{E}^-$ has an associated determinant bundle $\mathrm{Det}(\mathcal{E}) \equiv (\mathrm{Det}\,\mathcal{E}^+) \oplus \mathrm{Det}\,\mathcal{E}^-$. A connection $\nabla^{\mathcal{E}}$ on $\mathcal{E}$ with curvature $\Omega^{\mathcal{E}}$ induces a connection $\nabla^{\mathrm{Det}\,\mathcal{E}}$ on the determinant bundle, with curvature $\Omega^{\mathrm{Det}\,\mathcal{E}} = -\mathrm{str}(\Omega^{\mathcal{E}})$ equal to minus the first Chern form on the original bundle $\mathcal{E}$.

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In this paper, we investigate whether this property carries over to infinite rank bundles of physical interest. The immediate problem is that $\mathrm{str}(\Omega^{\mathcal{E}})$ involves a divergent sum. The paper breaks the problem down into two parts: (i) constructing the determinant bundle associated to an infinite rank superbundle, following [4,18]; (ii) defining the first Chern form of the superbundle, and relating it to the curvature on the determinant bundle.

As background, Quillen [18] constructed the determinant bundle with a natural metric associated to a family of Cauchy–Riemann operators on a Riemann surface, and computed its curvature. Later, Bismut and Freed [4] equipped the determinant bundle associated to a family of Dirac-type operators with a connection compatible with this Quillen metric, and computed the curvature in terms of local invariants of the underlying spin manifold. Freed [8] considered characteristic forms on loop groups, overcoming divergence problems via an ad hoc summation technique. In [2,15], more natural (but less tractable) heat kernel and zeta function regularization techniques were used to renormalize divergent expressions.

While these constructions involve no regularization in finite dimensions and hence are compatible, the regularization techniques introduce unavoidable discrepancies measured by Wodzicki residues in infinite dimensions. The choice of technique depends on the physical problem at hand. The Bismut–Freed definition of the regularized first Chern form is the curvature of the Bismut–Freed connection, which characterizes the local geometric obstruction to trivializing the determinant bundle, the (local geometric) anomaly. In contrast, our definition of the regularized first Chern form differs from the Bismut–Freed one by a Wodzicki residue. However, our regularization applies to a larger class of infinite dimensional bundles, such as the tangent bundle to loop groups and other infinite dimensional manifolds, and may lead to a theory of characteristic classes in infinite dimensions generalizing [8].

In more detail, in Sections 2–4, we formalize the construction of Quillen–Bismut–Freed determinant bundles in terms of determinant bundles associated to "half-weighted super vector bundles". We first restrict ourselves to a class of super vector bundles $\mathcal{E} \equiv \mathcal{E}^+ \oplus \mathcal{E}^-$, where $\mathcal{E}^\pm$ are vector bundles with fibers modeled on Sobolev spaces $H^{s^\pm}(M, E^\pm)$ of sections of some finite rank vector bundles $E^\pm$ over a closed Riemannian manifold M. A *half-weighted vector bundle* is such a superbundle together with a field/family

$$L \equiv \begin{bmatrix} 0 & L^- \\ L^+ & 0 \end{bmatrix}$$

of odd operators locally given by constant order elliptic operators (satisfying a common Agmon–Nirenberg condition) acting on smooth sections of $E \equiv E^+ \oplus E^-$. This local characterization makes sense globally if the transition maps are themselves zero order, grading preserving elliptic operators on M. If $\mathcal{E}$ comes with a Hermitian structure, as in our main example of families of Dirac operators, we will demand that L be self-adjoint. To a half-weighted super vector bundle $(\mathcal{E}, L)$ we associate a determinant bundle $\mathrm{Det}(\mathcal{E}, L)$, the Quillen determinant bundle of the family L^+.

Given a half-weighted vector bundle $(\mathcal{E}, L)$, we have a family $Q \equiv L^2 = L^-L^+ \oplus L^+L^-$ of positive, self-adjoint, locally elliptic operators acting fiberwise on $\mathcal{E}$. As in [16], we call $(\mathcal{E}, Q)$ a *weighted vector bundle*. The *weight* Q can be viewed as metric data on the infinite dimensional vector bundle $\mathcal{E}$, and the existence of L allows us to view $\mathcal{E}$ as a spinor bundle with Clifford multiplication given by the *half weight* L.

Starting in Section 5, we construct regularized first Chern forms. Using Q, we define *Q-weighted traces* tr^Q and *Q-weighted supertraces* str^Q, which are linear functionals on sections of $CL(\mathcal{E})$, the bundle of operators which are locally given by classical pseudo-differential operators (PDOs) on the fibers of $\mathcal{E}$. We define the *weighted first Chern form* of a superconnection $\nabla^{\mathcal{E}}$ on $(\mathcal{E}, Q)$ as the Q-weighted supertrace $\mathrm{str}^Q(\Omega^{\mathcal{E}})$ of the curvature of the connection, provided $\Omega^{\mathcal{E}}$ is a two-form with values in PDOs on the fibers of $\mathcal{E}$.

Our main results (Section 7, Theorems 3 and 5) show that the curvature of the Bismut–Freed connection on the determinant bundle associated to a half-weighted superbundle with connection differs from (minus) the weighted first Chern form on the superbundle by a linear combination of Wodzicki residues (Theorem 3) or equivalently by a renormalized trace farm (Theorem 5). This obstruction to the finite dimensional formula arises from the non-vanishing of $[\nabla^{\mathcal{E}}, \mathrm{str}^Q]$, a feature of the infinite dimensional weighting procedure. We express this obstruction in two ways:

- via zeta function regularization, using weighted supertraces and evaluating the obstruction $[\nabla^{\mathcal{E}}, \mathrm{tr}^Q]$ in terms of a Wodzicki residue (Theorem 3);
- via heat kernel regularization, using a one-parameter family of Bismut connections [3], and evaluating the obstruction in terms of regularized trace farms (Theorem 5).

We also show (Corollary 6) that the weighted first Chern form is more local than the curvature of the Bismut–Freed connection in a certain technical sense. In the proof of the Corollary, we see that the curvature of the superbundle is a multiplication operator and, therefore, not trace-class. Thus regularization procedures are necessary to define the first Chern form.

In Section 8, we illustrate the main results with a string theory/Teichmüller theory example. Here the action of H^{s+1} diffeomorphisms of a closed surface Λ on the manifold $\mathcal{A}(\Lambda)$ of smooth almost complex structures on Λ gives rise to a family $\alpha_J : H^{s+1}(T\Lambda) \to H^s(T_1^1\Lambda)$, $J \in \mathcal{A}(\Lambda)$ of elliptic operators. Setting $\mathcal{E}^+ \equiv T\mathcal{A}^s(\Lambda)|_{\mathcal{A}(\Lambda)}$ and $\mathcal{E}^- \equiv \mathcal{A}(\Lambda) \times H^s(T_1^1\Lambda)$, we can view

$$\left(\mathcal{E} \equiv \mathcal{E}^+ \oplus \mathcal{E}^-, L \equiv \begin{bmatrix} 0 & \alpha^* \\ \alpha & 0 \end{bmatrix}\right)$$

as a half-weighted superbundle. We identify the Bismut–Freed superconnection with classical connections in Teichmüller theory.

In Appendix A, we collect some superconnection calculations. In Appendix B, as suggested by the different proofs of Theorems 3 and 5, we relate Wodzicki residues to the trace forms of [10].

1.1. Notation

Let E be a finite rank Hermitian or Riemannian vector bundle over a Riemannian manifold M. The natural L^2 inner product on the smooth sections of E is defined by

$$\langle \sigma, \tau \rangle \equiv \int_M \langle \sigma(x), \tau(x) \rangle_x \, \mathrm{d}\mu(x),$$

where μ is the volume measure on M and $\langle\cdot,\cdot\rangle_x$, the inner product on the fiber of E above x.

We denote by $CL(M, E)$ the algebra of classical PDOs acting on smooth sections of E, by $\mathrm{Ell}(M, E)$ the multiplicative subset of elliptic PDOs, by $\mathrm{Ell}^{\mathrm{sa}}(M, E)$ the subset of self-adjoint elliptic PDOs and by $\mathrm{Ell}^{+}(M, E)$ the subset of positive elliptic PDOs. Adding the subscript ord > 0 to these sets restricts to operators of strictly positive order. Adding the superscript $*$ restricts to injective operators.

In the following, we take $s > (\dim M)/2$. Recall that for $s > (\dim M)/2$, we have $H^{k+s}(M, E) \subset C^k(M, E)$ for any $k \in \mathbf{N}$, where $H^t(M, E)$ (respectively $C^k(M, E)$) denotes the space of H^t (respectively C^k) sections of the bundle E.

2. A class of vector bundles

We say that a Hilbert space H lies in the class $\mathcal{CH}$ if there is a closed smooth Riemannian manifold M, a finite rank Hermitian/Riemannian vector bundle E over M, and $s > (\dim M)/2$ such that $H = H^s(M, E)$. For example, for G be a Lie group and $\mathrm{Lie}(G)$ its Lie algebra, the Lie algebra $H^s(M, \mathrm{Lie}(G))$ of the Hilbert current group $H^s(M, G)$ lies in $\mathcal{CH}$.

Let $\mathcal{CE}$ be the class of Riemannian Hilbert vector bundles $\mathcal{E} \to X$ over a (possibly infinite dimensional) manifold X with fibers modeled on a separable Hilbert space $H = H^s(M, E)$ in $\mathcal{CH}$ and with transition maps in $CL(M, E)$. Note that these PDOs have coefficients only in some Sobolev class. However, the PDOs in the examples below are locally given by multiplication operators, and are as tractable as PDOs with smooth coefficients.

$\mathcal{CX}$ denotes the class of infinite dimensional manifolds X with tangent bundle TX in $\mathcal{CE}$. Since the transition maps are bounded, they correspond to operators of order zero. Moreover, the transition maps are invertible, so they in fact lie in $\mathrm{Ell}(M, E)$.

We now give examples of manifolds in $\mathcal{CX}$ and vector bundles in $\mathcal{CE}$.

Examples. (i) Finite rank vector bundles lie in $\mathcal{CE}$. To see this, we take as base manifold a point $\{*\}$, and as the bundle E the trivial bundle $\{*\} \times \mathbf{R}^d$ (or $\{*\} \times \mathbf{C}^d$ if the bundle is complex). The transition functions belong to $\mathrm{Ell}(\{*\}, E) = \mathrm{Gl}_d(\mathbf{R})$ (or $\mathrm{Gl}_d(\mathbf{C})$). We say that M is reduced to a point.

(ii) If G is a Lie group and $s > (\dim M)/2$, the current group $H^s(M, G)$ is a Hilbert Lie group having a left invariant atlas $\phi_\gamma(u)(x) \equiv \exp_{\gamma(x)}(u(x))$, for $x \in M$, $\gamma \in H^s(M, G)$, where $\exp_{\gamma(x)}$ is the exponential coordinate chart at $\gamma(x)$ induced by a left invariant Riemannian metric on G. The transition functions are given by multiplication operators, which indeed are PDOs.

(iii) Let $M \equiv \Lambda$ be a closed, oriented, Riemannian surface of genus $p > 1$, and let $\mathcal{A}^s(\Lambda)$, $s > 1$, be the space of almost complex structures on Λ of Sobolev class H^s, i.e.

$$\mathcal{A}^s(\Lambda) = \{J \in H^s(T_1^1\Lambda),\ J_x^2 = -\mathrm{Id}_x,\ J_x \text{ preserves orientation of } T_x\Lambda \text{ for } x \in \Lambda\}.$$

$\mathcal{A}^s(\Lambda)$ is a smooth Hilbert manifold with tangent space at $J \in \mathcal{A}^s(\Lambda)$ given by [20]

$$T_J\mathcal{A}^s(\Lambda) = \{H \in H^s(T_1^1\Lambda), HJ + JH = 0\}.$$

(The set of smooth almost complex structures $\mathcal{A}(\Lambda) = \cap_{s>1}\mathcal{A}^s(\Lambda)$ is only a Fréchet manifold). We determine the transition maps. The charts are given pointwise by the matrix exponential map $\exp_J H(x) \equiv \exp_{J(x)} H(x)$. Hence the transition maps as maps on $H^s(T_1^1\Lambda)$ are multiplication operators, so they are PDOs of order zero. Thus $T\mathcal{A}^s(\Lambda)$ is in $\mathcal{CE}$ with fibers modeled on $H^s(\Lambda, E)$ where $E \equiv T_1^1\Lambda$. In the string theory example in Appendix B, we consider the subbundle given by restricting $T\mathcal{A}^s(\Lambda)$ to the manifold $\mathcal{A}(\Lambda)$:

$$\mathcal{E}^- \equiv T\mathcal{A}^s(\Lambda)|_{\mathcal{A}(\Lambda)}. \tag{2.1}$$

$\mathcal{E}^-$ has an almost complex structure defined fiberwise by

$$\mathcal{J}^-(J)(H) \equiv J \cdot H,$$

where '·' denotes pointwise matrix multiplication. Notice that if J is smooth and H of class H^s, then JH is of class H^s. $\mathcal{J}^-$ induces a splitting

$$\mathcal{E}^- \equiv \mathcal{E}^{-1,0} \oplus \mathcal{E}^{-0,1},$$

where the fibers of the subbundles above $J \in \mathcal{A}(\Lambda)$ are

$$\mathcal{E}_J^{-1,0} \equiv \mathrm{Ker}(\mathcal{J}_J^- - \mathrm{i}), \qquad \mathcal{E}_J^{-0,1} \equiv \mathrm{Ker}(\mathcal{J}_J^- + \mathrm{i}).$$

Because the almost complex structure is defined pointwise by $(J \cdot H)(x) = J(x)H(x)$ for $x \in \Lambda$ and hence defines a PDO, the transition functions of these subbundles are also given by PDOs. Thus $\mathcal{E}_J^{-1,0}, \mathcal{E}_J^{-0,1}$ lie in $\mathcal{CE}$.

(iv) In Appendix B, we also consider the trivial bundle

$$\mathcal{E}^+ \equiv \mathcal{A}(\Lambda) \times H^{s+1}(T\Lambda), \tag{2.2}$$

which clearly lies in $\mathcal{CE}$. $\mathcal{E}^+$ has a natural almost complex structure $\mathcal{J}^+$ defined fiberwise by the almost complex structure on the tangent space to Λ:

$$\mathcal{J}^+(J)u \equiv Ju.$$

With respect to the complex structure J, $T\Lambda$ splits into $T\Lambda = T^{1,0}\Lambda \oplus T^{0,1}\Lambda$, with $T^{1,0}\Lambda \equiv \mathrm{Ker}(J - \mathrm{i})$, $T^{0,1}\Lambda \equiv \mathrm{Ker}(J + \mathrm{i})$. $\mathcal{E}^+$ therefore splits into subbundles

$$\mathcal{E}^+ = \mathcal{E}^{+1,0} \oplus \mathcal{E}^{-1,0},$$

whose fibers above $J \in \mathcal{A}(\Lambda)$ are

$$\mathcal{E}_J^{+1,0} \equiv \mathrm{Ker}(\mathcal{J}_J^+ - \mathrm{i}) = H^{s+1}(\mathrm{Ker}(J - \mathrm{i})),$$
$$\mathcal{E}_J^{+0,1} \equiv \mathrm{Ker}(\mathcal{J}_J^+ + \mathrm{i}) = H^{s+1}(\mathrm{Ker}(J + \mathrm{i})).$$

3. Weighted vector bundles and half-weighted super vector bundles

A *weighted Hilbert space* is a pair (H, Q) with H in $\mathcal{CH}$ and $Q \in \mathrm{Ell}^+_{\mathrm{ord}>0}(M, E)$.

3.1. Bundles of elliptic operators

Let $\mathcal{E}$ be a vector bundle in $\mathcal{CE}$ over a manifold X with $\mathcal{E}$ modeled on a separable Hilbert space H. For $x \in X$, let $CL(\mathcal{E}_x)$ be the set of operators A_x acting densely on the fiber $\mathcal{E}_x$ above x such that for any local trivialization $\phi : \mathcal{E}|_{U_x} \to U_x \times H$ near x, the operator $\phi^\sharp A(x) \equiv \phi(x)A_x\phi(x)^{-1}$ lies in $CL(M, E)$. Here $\phi(x) : \mathcal{E}_x \to H$ is the isomorphism induced by the trivialization. Similarly, let $\mathrm{Ell}(\mathcal{E}_x)$ be the set of operators A_x acting densely on T_xX such that for any local trivialization $\phi : \mathcal{E}|_{U_x} \to U_x \times H$ near x, the operator $\phi^\sharp A$ lies in $\mathrm{Ell}(M, E)$. From this point on we will omit the subscript x.

These definitions are independent of the choice of local chart. Indeed, since transition functions are given by operators in the group $CL^*(M, E)$ of invertible elements in $CL(M, E)$, the condition $\phi^\sharp A \in CL(M, E)$ is independent of the choice of ϕ. Since the principal symbol is multiplicative and since ellipticity is characterized by invertibility of the principal symbol, the condition $\phi^\sharp A(x) \in \mathrm{Ell}(M, E)$ is also independent of the choice of ϕ. Notice that the order of $\phi^\sharp A$ is independent of the choice of local chart, so we can speak of the order of A. This gives rise to bundles $CL(\mathcal{E})$ and $\mathrm{Ell}(\mathcal{E})$ with fiber at given x by $CL(\mathcal{E}_x)$ and $\mathrm{Ell}(\mathcal{E}_x)$, respectively. In particular, a section of the second bundle is a family of elliptic operators parameterized by the base.

3.2. Weighted bundles

A local section Q of $\mathrm{Ell}(\mathcal{E})$, with $\mathcal{E}$ modeled on some $H^s(M, E)$, is *positive self-adjoint* if for all x in the support of Q, and in any local chart (U, ϕ) around x, the operator $\phi^\sharp Q(x)$ lies in $\mathrm{Ell}^+(M, E)$. A *weighted bundle* is a pair $(\mathcal{E}, Q)$ with $\mathcal{E}$ in $\mathcal{CE}$ and Q a section of positive self-adjoint operators of constant order in $\mathrm{Ell}(\mathcal{E})$. A *weighted manifold* (X, Q) is a manifold in $\mathcal{CX}$ such that (TX, Q) is a weighted vector bundle. The operator $\phi^\sharp Q$ is by definition a *weight* on the model space H of X.

If $\mathcal{E}$ has no Hermitian structure, we can relax our definition of weight to be those Q which are locally elliptic of constant order with the Agmon–Nirenberg condition (i.e. with leading symbol having all eigenvalues lying outside some common fixed angle at the origin). In this generality, the choice of a weight Q replaces the structure group $GL(H) = GL(H^s(M, E))$ of $\mathcal{E}$ by the subgroup $CL_0^*(M, E) = CL^*(M, E) \cap GL(H)$. (This is not a classical reduction of structure group, since $CL_0^*(M, E)$ is not a Lie subgroup of $GL(H)$ in the standard topologies.) Putting a Hermitian structure on $\mathcal{E}$ is equivalent to a true reduction of $CL_0^*(M, E)$ to $CL_0^*(M, E) \cap U(H)$.

Examples. We return to examples (i)–(iv).

(i) When $\mathcal{E}$ is a rank n bundle, we can view it as before as a bundle of sections over a manifold reduced to a point. Then $GL(H) = CL_0^*(M, E) = CL^*(M, E) = GL(n, \mathbf{C})$, so a choice of weight is irrelevant. However, $CL^*(M, E) \cap U(H) = U(n)$, so the "true reduction" amounts to a choice of Hermitian metric.

(ii) For the current groups $H^s(M, G)$, let $Q_0 \equiv \Delta \oplus 1_{\mathrm{Lie}(G)}$ be the Laplace–Beltrami operator on M with values in the Lie algebra $\mathrm{Lie}(G)$ of the group G, for Δ the Laplace–Beltrami operator acting on complex valued functions on M. For $\gamma \in H^s(M, G)$, setting $Q(\gamma) \equiv L_\gamma^{-1} Q_0 L_\gamma$, where L_γ is left multiplication by γ, yields a weighted manifold $(H^s(M, G), Q)$.

(iii) and (iv) We consider the bundles $\mathcal{E}^{\pm}$ defined above. For $J \in \mathcal{A}(\Lambda)$, let $\alpha_J : H^{s+1}(T\Lambda) \to H^s(T_1^1\Lambda)$ be the operator defined by the Lie derivative of J

$$\alpha_J u = \frac{\mathrm{d}}{\mathrm{d}t}(f_{u,t}^* J), \tag{3.1}$$

where $f_{u,t}$ is the flow of the vector field u. α_J is a first-order elliptic operator with range $T_J\mathcal{A}^s(\Lambda) = \{H \in H^s(T_1^1\Lambda), HJ + JH = 0\}$ [20]. Its adjoint α_J^* is defined with respect to the Hermitian products:

$$\langle u, v\rangle_J^+ \equiv \int_\Lambda \mathrm{d}\mu_J(x)\langle u, v\rangle_{g_J}, \tag{3.2a}$$

$$\langle H, K\rangle_J^- \equiv \int_\Lambda \mathrm{d}\mu_J(x)\langle H, K\rangle_{g_J}. \tag{3.2b}$$

Here g_J is the unique metric of constant curvature -1 among the conformal class of metrics for which J is orthogonal [20], and $\mathrm{d}\mu_J$ is the associated volume form. Note that $\langle H, K\rangle = \mathrm{tr}(HK^*)$, where K^* is the Hermitian adjoint of the matrix representing the (1, 1) tensor K with respect to g_J. Since $\alpha_J^*\alpha_J$ and $\alpha_J\alpha_J^*$ are elliptic, the families

$$Q^+ \equiv \{Q_J^+ \equiv \alpha_J^*\alpha_J,\ J \in \mathcal{A}(\Lambda)\}, \qquad Q^- \equiv \{Q_J^- \equiv \alpha_J\alpha_J^*,\ J \in \mathcal{A}(\Lambda)\},$$

yield weighted bundles $(\mathcal{E}^+, Q^+)$ and $(\mathcal{E}^-, Q^-)$, respectively. Thus we get a weighted super vector bundle

$$\mathcal{E} = \mathcal{E}^+ \oplus \mathcal{E}^-, \qquad Q \equiv Q^+ \oplus Q^-. \tag{3.3}$$

3.3. *Half-weighted super vector bundles*

For a super vector bundle $\mathcal{E}$ in $\mathcal{CE}$ with fibers modeled on some $H^{s^+}(E^+) \oplus H^{s^-}(E^-)$, $s^{\pm} > (\dim M)/2$, via local charts we can write a local section L of $\mathrm{Ell}(\mathcal{E})$ in matrix form

$$L = \begin{bmatrix} L_{++} & L_{+-} \\ L_{-+} & L_{--} \end{bmatrix}.$$

Provided the transition maps are even, it makes sense to consider the class of odd operators, i.e. those which locally have only off-diagonal terms. We define a *half-weighted superbundle* to be a pair $(\mathcal{E}, L)$, where $\mathcal{E}$ is a superbundle in $\mathcal{CE}$ with even transition maps and L is a section of odd self-adjoint operators in $\mathrm{Ell}(\mathcal{E})$ of non-zero order.

To a half-weighted superbundle $(\mathcal{E}, L)$, we can associate a weighted superbundle $(\mathcal{E}, Q \equiv L^2)$. Since L is odd, we can write

$$L \equiv \begin{bmatrix} 0 & L^- \equiv (L^+)^* \\ L^+ & 0 \end{bmatrix},$$

so the weight Q can be written as

$$Q = Q^+ \oplus Q^- \equiv L^-L^+ \oplus L^+L^- = (L^+)^*L^+ \oplus (L^-)^*L^-.$$

Examples.

$$\left(\mathcal{E} \equiv \mathcal{E}^+ \oplus \mathcal{E}^-, \qquad L \equiv \left\{\begin{bmatrix} 0 & \alpha_J^* \\ \alpha_J & 0 \end{bmatrix}, \quad J \in \mathcal{A}(\Lambda)\right\}\right),$$

with $\mathcal{E}^\pm$ as in (2.1) and (2.2), and α_J as in (3.1), is a half-weighted super vector bundle. If α_J stabilizes the fiber $\mathcal{E}_J^{1,0}$ for each $J \in \mathcal{A}(\Lambda)$, we can build a complex half-weighted bundle $(\mathcal{E}^{1,0}, L^{1,0} \equiv \{L_J^{1,0}, J \in \mathcal{A}(\Lambda)\})$, where $L_J^{1,0} = \alpha_J|_{\mathcal{E}_J^{1,0}}$.

As shown in Lemma 7, $L_J^{1,0}$ is a Cauchy–Riemann operator, the historically first case of examples provided by spinor bundles on even dimensional manifolds [4,18]. Let $\pi : Z \to B$ be a smooth fibration of even dimensional spin manifolds $\{M_b, b \in B\}$, and let $\mathcal{E} \to B$ be an infinite dimensional super vector bundle with fiber $H^s(M_b, E_b)$ for a smooth family $\{E_b, b \in B\}$ of Clifford bundles on M_b. The Dirac operators $D_b = D_b^+ \oplus D_b^-$ act on $H^s(M_b, E_b)$ as elliptic operators. For

$$L_b \equiv \begin{bmatrix} 0 & D_b^- = (D_b^{+*}) \\ D_b^+ & 0 \end{bmatrix}, \tag{3.4}$$

$(\mathcal{E}, L)$ is a half-weighted superbundle.

3.4. *From group actions to half-weighted superbundles*

Half-weighted superbundles also arise from group actions. Let $\mathcal{G}$ and $\mathcal{P}$ be two infinite dimensional Hilbert manifolds modeled, respectively, on $H^{s^+}(M, E^+)$ and $H^{s^-}(M, E^-)$, where $E = E^+ \oplus E^-$ is a superbundle over M, such that

(a) $\mathcal{G}$ has a smooth group multiplication on the right: $R_{\gamma_0} : \mathcal{G} \to \mathcal{G}, \gamma \mapsto \gamma\gamma_0$ for $\gamma_0 \in \mathcal{G}$.
(b) $\mathcal{G}$ acts on $\mathcal{P}$ on the right by $\Theta : \mathcal{G} \times \mathcal{P} \to \mathcal{P}, (\gamma, p) \to p \cdot \gamma$, inducing a smooth map $\theta_p : \mathcal{G} \to \mathcal{P}, \gamma \to p \cdot \gamma$ for $p \in \mathcal{P}$.
(c) The differential $\alpha_p \equiv \mathrm{d}\theta_p : T_e\mathcal{G} \to T_p(\mathcal{P})$ is elliptic, with order independent of p.

Let $\mathcal{E}^+ \equiv B \times \mathrm{Lie}(\mathcal{G})$, where B is a submanifold of $\mathcal{P}$, $\mathrm{Lie}(\mathcal{G}) = T_e\mathcal{G}$, and $\mathcal{E}^- = T\mathcal{P}|_B$. Then

$$\left(\mathcal{E} = \mathcal{E}^+ \oplus \mathcal{E}^-, \qquad \mathcal{L} = \left\{L_b \equiv \begin{bmatrix} 0 & \alpha_b^* \\ \alpha_b & 0 \end{bmatrix}, \quad b \in B\right\}\right)$$

is a half-weighted superbundle.

Example. In Example (iii) of Section 1.1 above, let $\mathcal{G} \equiv \mathrm{Diff}_0^{s+1}(\Lambda)$ be the group of isotopies (i.e. diffeomorphisms homotopic to the identity) of Λ of Sobolev class H^{s+1}. Although $\mathcal{G}$ is not a Lie group, it is a Hilbert manifold modeled on $H^{s+1}(T\Lambda)$ with a smooth multiplication on the right. $\mathcal{G}$ acts on $\mathcal{A}^s(\Lambda)$ (which we recall is modeled on $H^s(T_1^1)$ by pullback, and this action satisfies (a) and (b) above (see [20]). Since α_j in (3.1) is elliptic, the family

$$L \equiv \left\{ \begin{bmatrix} 0 & \alpha_J^* \\ \alpha_J & 0 \end{bmatrix}, \quad J \in \mathcal{A} \right\}$$

yields a half-weighted structure on the bundle $\mathcal{E}$ in (3.3).

4. From a half-weighted super vector bundle to the determinant bundle

Let $(\mathcal{E}, L)$ be a half-weighted super vector bundle over a manifold B, which as above determines the weighted superbundle $(\mathcal{E}, Q)$ with $Q = L^2$. From $(\mathcal{E}, L)$, we construct the *determinant bundle* $\mathrm{Det}(\mathcal{E}, L) = \mathrm{Det}(\mathcal{E})$, following [4,5,18].

As before, set $\mathcal{E} = \mathcal{E}^+ \oplus \mathcal{E}^-$, where $\mathcal{E}^\pm$ has fibers modeled on $H^{s^\pm}(M, E^\pm)$, and write a section L of $\mathrm{Ell}(\mathcal{E})$ consisting of odd, self-adjoint operators in the form

$$\begin{bmatrix} 0 & L^- = (L^+)^* \\ L^+ & 0 \end{bmatrix}.$$

Let m be the order of L_b, $b \in B$, and set $s_+ = s_- + m$. This yields a family of Fredholm operators $L_b^+ : H^{s^+}(M, E^+) \to H^{s^-}(M, E^-)$. As Quillen shows, there is a line bundle, the determinant bundle $\mathrm{Det}(\mathcal{E})$ over B, with fiber $\mathrm{Det}(\mathcal{E})_b \simeq (\Lambda^{\mathrm{top}} \operatorname{Ker} L_b^+)^* \oplus \Lambda^{\mathrm{top}} \operatorname{Coker} L_b^+$, where Λ^{top} denotes the top exterior power. $\mathrm{Det}(\mathcal{E})$ has a canonical section $\mathrm{Det}\, L^+(b)$ given by $\alpha(b)(e_1 \wedge \cdots \wedge e_n)^* \otimes (f_1 \wedge \cdots \wedge f_m)$, where $\{e_i\}$, respectively $\{f_j\}$, are orthonormal bases of the eigenvalues of $L_b^- L_b^+$, respectively $L_b^+ L_b^-$, lying below some $a \in \mathbf{R}$ not in the spectrum of either operator, and $\alpha(b)$ is the determinant of the matrix of L_b^+ with respect to the bases $\{e_i\}$, $\{f_j\}$; equivalently, $L_b^+ e_1 \wedge \cdots \wedge L_b^+ e_n = \alpha(b) f_1 \wedge \cdots \wedge f_n$. Note that $\mathrm{Det}\, L^+$ is zero iff L_b^+ is non-invertible.

4.1. *A family of connections on the determinant bundle*

Fix $\varepsilon > 0$. At any point $b \in B$, where L_b is injective, the ε-cutoff determinant of the self-adjoint elliptic operator $Q_b^+ = L_b^- L_b^+$ is defined by

$$\det_\varepsilon Q_b^+ \equiv \exp\left[-\int_\varepsilon^\infty \frac{1}{t} \operatorname{tr}(\mathrm{e}^{-tQ_b^+})\, \mathrm{d}t \right].$$

For non-invertible L_b^+, we subtract off the dimension of the zero eigenspace before taking the trace in the integral. The cutoff determinants yield a one-parameter family of Quillen metrics $\{\|\cdot\|_{Q,\varepsilon}, \varepsilon > 0\}$ on $\mathrm{Det}(\mathcal{E})$ defined by

$$\|\mathrm{Det}\, L_b^+\|_{Q,\varepsilon}^2 \equiv \begin{cases} \det_\varepsilon Q_b^+, & L_b^+ \text{ invertible;} \\ \det_\varepsilon Q_{(b,>a)}^+ \cdot \det Q_{(b,<a)}^+, & L_b^+ \text{ non-invertible.} \end{cases}$$

Here $Q_{(b,>a)}^+$ is the restriction of Q_b^+ to the eigenspaces above a, with determinant computed as for Q_b^+, and $Q_{(b,<a)}^+$ is the restriction of Q_b^+ to the finite dimensional eigenspaces below

a, with the determinant computed as usual. Since our formulas involve regularization only on the eigenspaces above a fixed, locally defined a, for notational simplicity we will assume from now on that $L_b^{\pm}$ is invertible. In general, our formulas can be modified via the cutoff operator $Q^+_{(b,>a)}$.

Given a connection $\nabla^{\mathcal{E}}$ on $\mathcal{E}$, as in [4] we can define a one-parameter family $\{\nabla^{\mathrm{Det}(\mathcal{E}),\varepsilon}, \varepsilon > 0\}$ of connections on $\mathrm{Det}(\mathcal{E})$ compatible with the metrics $\{\|\cdot\|_{Q,\varepsilon}, \varepsilon > 0\}$ by

$$\begin{aligned}(\mathrm{Det}\, L_b^+)^{-1}\nabla^{\mathrm{Det}(\mathcal{E}),\varepsilon}\,\mathrm{Det}\, L_b^+ &\equiv \mathrm{tr}((L_b^+)^{-1}\nabla^{\mathrm{Hom}(\mathcal{E})}L_b^+\,\mathrm{e}^{-\varepsilon Q_b^+})\\ &= \tfrac{1}{2}(\mathrm{dlog}\,\mathrm{det}_\varepsilon Q_b^+ + \mathrm{str}((L_b)^{-1}\nabla^{\mathrm{Hom}(\mathcal{E})}L_b\,\mathrm{e}^{-\varepsilon Q_b})).\end{aligned} \tag{4.1}$$

Here $\nabla^{\mathrm{Hom}}(\mathcal{E})$ denotes the connection on $\mathrm{Hom}(\mathcal{E}^+, \mathcal{E}^-)$ induced by $\nabla^{\mathcal{E}}$, and str denotes the supertrace, defined by

$$\mathrm{str}(A) = \mathrm{str}\begin{bmatrix} A^+ & X \\ Y & A^- \end{bmatrix} \equiv \mathrm{tr}(A^+) - \mathrm{tr}(A^-). \tag{4.2}$$

This definition is motivated by the corresponding formula for the natural connection (6.3) on the determinant line bundle for a finite rank superbundle.

4.2. Renormalized limits

Following [5, Chapter 9], from the family of connections $\{\nabla^{\mathrm{Det}(\mathcal{E}),\varepsilon}, \varepsilon > 0\}$, we build a renormalized connection by taking a renormalized limit as $\varepsilon \to 0$. More precisely, for $(m, n) \in (\mathbf{N} \setminus \{0\}) \times \mathbf{N}$, $\alpha \in \mathbf{R}$, let $\mathcal{F}_{m,n,\alpha}$ be the set of functions $f : \mathbf{R}^+ \setminus \{0\} \to \mathbf{C}$ such that there exist $a_j, b_j, c_j \in \mathbf{C}$ with

$$f(\varepsilon) \sim \sum_{j=0}^{\infty} a_j\varepsilon^{\lambda_j} + \sum_{j=0,\lambda_j\in\mathbf{Z}}^{\infty} b_j\varepsilon^{\lambda_j}\log\varepsilon + \sum_{j=0}^{\infty} c_j\varepsilon^j,$$

as $\varepsilon \to 0$, where $\lambda_j \equiv (j-\alpha-n)/m$. In other words, for $J \in \mathbf{N}$ and $K_J \equiv [\alpha]+mJ+n \in \mathbf{N}$, we have

$$f(\varepsilon) = \sum_{j=0}^{K_J} a_j\varepsilon^{\lambda_j} + \sum_{j=0,\lambda_j\in\mathbf{Z}}^{K_J} b_j\varepsilon^{\lambda_j}\log\varepsilon + \sum_{j=0}^{J} c_j\varepsilon^j + \mathrm{o}(\varepsilon^J)$$

(cf. (5.3)). If $\alpha \in \mathbf{Z}$, there is a redundancy since constant terms can arise in the first and last sum. We call such a function *renormalizable*, and for $f \in \mathcal{F}_m \equiv \cup_{n\in\mathbf{N},\alpha\in\mathbf{R}}\mathcal{F}_{m,n,\alpha}$, we define *$\mu$-renormalized limit* of f at zero by

$$\mathrm{Lim}^{\mu}_{\varepsilon\to 0} f(\varepsilon) = a_{\alpha+n} + c_0 - \mu b_{\alpha+n}, \tag{4.3}$$

where we set $a_{\alpha+n} = 0$ and $b_{\alpha+n} = 0$ if $\alpha + n \notin \mathbf{N}$. Thus $\mathrm{Lim}^{\mu}_{\varepsilon\to 0} f(\varepsilon)$ is the constant term in f's asymptotic expansion minus μ times the coefficient of $\log\varepsilon$ ([5] only considers the case $\mu = \gamma$, the Euler constant). If there is no logarithmic divergence, then $\mathrm{Lim} \equiv \mathrm{Lim}^{\mu} \equiv \mathrm{Lim}^{\mu}_{\varepsilon\to 0}$ is independent of μ.

4.3. A renormalized connection on the determinant bundle

As in [4,5], we set

$$\begin{aligned}&(\mathrm{Det}\, L_b^+)^{-1}\nabla^{\mathrm{Det},\mu}\, \mathrm{Det}\, L_b^+\\ &\quad\equiv \mathrm{Lim}^{\mu}_{\varepsilon\to 0}\, \mathrm{tr}((L_b^+)^{-1}(\nabla^{\mathrm{Hom}(\mathcal{E})}L_b^+)\mathrm{e}^{-\varepsilon L_b^- L_b^+})\\ &\quad= \tfrac{1}{2}(\mathrm{dlog}\,\det{}_\mu Q_b^+ + \mathrm{Lim}^{\mu}_{\varepsilon\to 0}\,\mathrm{str}((L_b)^{-1}(\nabla^{\mathrm{Hom}(\mathcal{E})}L_b)\mathrm{e}^{-\varepsilon Q_b})),\end{aligned}$$

where

$$\det{}_\mu Q_b^+ \equiv \exp\left(-\mathrm{Lim}^{\mu}_{\varepsilon\to 0}\int_\varepsilon^\infty \frac{1}{t}\,\mathrm{tr}(\mathrm{e}^{-tQ_b^+})\,\mathrm{d}t\right).$$

The renormalized connection $\nabla^{\mathrm{Det}(\mathcal{E}),\mu} = \nabla^{\mathrm{Det},\mu}$ is compatible with the renormalized Quillen metric given by

$$\|\mathrm{Det}\, L_b^+\|_{Q,\mu} \equiv \sqrt{\det{}_\mu Q_b^+}.$$

The curvature of $\nabla^{\mathrm{Det},\mu}$ is denoted by $\Omega^{\mathrm{Det},\mu}$.

5. First Chern forms on weighted vector bundles

The first Chern form on a finite rank Hermitian bundle with connection is the trace of the curvature. In infinite rank, one cannot expect curvature to be trace-class in general, so we need to regularize (or renormalize) the trace. We will use extra data of the weights of Section 3 to define weighted traces in two steps: (i) defining a one-parameter family of weighted traces; (ii) taking a renormalized limit.

Let $(\mathcal{E}, Q)$ be a weighted vector bundle in $\mathcal{CH}$ with fibers modeled on $H^s(M, E)$, and let A be a section of $CL(\mathcal{E})$. Q is positive elliptic with strictly positive order, so for $\varepsilon > 0$, $\mathrm{e}^{-\varepsilon Q}$ is infinitely smoothing when seen in a local chart. Thus $A\,\mathrm{e}^{-\varepsilon Q}$ is trace-class when considered in a local chart as a trace-class operator acting on $L^2(M, E)$. We remark that a trace-class operator for the L^2 inner product $\langle\cdot,\cdot\rangle$ can be considered equally well as a trace-class operator with respect to the H^s scalar product

$$\langle\sigma, \rho\rangle^s \equiv \langle (Q+1)^{s/\mathrm{ord}(Q)}\sigma, (Q+1)^{s/\mathrm{ord}(Q)}\rho\rangle.$$

5.1. A family of weighted pseudo-traces

We define a one-parameter family of Q-*pseudo-traces* of A by

$$\mathrm{tr}^Q_\varepsilon(A) \equiv \mathrm{tr}(A\,\mathrm{e}^{-\varepsilon Q}), \tag{5.1}$$

for $\varepsilon > 0$. Again this definition should be understood in a local chart, but it is independent of the choice of chart, since for an invertible operator C, we have $\mathrm{tr}(CAC^{-1}\,\mathrm{e}^{-\varepsilon CQC^{-1}}) = \mathrm{tr}(CA\,\mathrm{e}^{-\varepsilon Q}C^{-1}) = \mathrm{tr}(A\,\mathrm{e}^{-\varepsilon Q})$. We emphasize that pseudo-traces are not traces in the usual sense. First, $\mathrm{tr}_\varepsilon^Q[A_1, A_2] \neq 0$ in general. Moreover, unlike the finite dimensional case, if $\{Q_t, t \in \mathbf{R}\}$, is a one-parameter family of weights and $\{A_t, t \in \mathbf{R}\}$ is a one-parameter family of PDOs, then for fixed $\varepsilon > 0$

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathrm{tr}_\varepsilon^{Q_t}(A_t) \neq \mathrm{tr}_\varepsilon^{Q_0}(\dot{A}) - \varepsilon\, \mathrm{tr}_\varepsilon^{Q_0}(A_0 \dot{Q}),$$

where $\dot{T} \equiv (\mathrm{d}/\mathrm{d}t)|_{t=0} T_t$, as one would expect from a formal differentiation of (5.1), since in general neither $\dot{Q}$ nor A_0 commutes with Q_0. If either $\dot{Q}$ or A_0 commutes with Q_0, this equation holds by [9, Section 1.9]. These obstructions can be analyzed more carefully using the renormalized pseudo-traces in the next paragraph (see [6,16]).

By (5.1), the one-parameter family of connections on the determinant bundle given by (4.1) is

$$\begin{aligned}(\mathrm{Det}_L^+)^{-1} \nabla^{\mathrm{Det}\,\mathcal{E},\varepsilon}\, \mathrm{Det}\, L^+ &= \mathrm{tr}_\varepsilon^{Q^+}((L^+)^{-1} \nabla^{\mathrm{Hom}(\mathcal{E})} L^+) \\ &= \tfrac{1}{2}(\mathrm{dlog}\,\mathrm{det}_\varepsilon Q^+ + \mathrm{str}_\varepsilon^Q (L^{-1} \nabla^{\mathrm{Hom}(\mathcal{E})} L)).\end{aligned} \tag{5.2}$$

As before $Q^\pm \equiv L^\mp L^\pm$ and $Q \equiv Q^+ \oplus Q^-$.

5.2. *Renormalized pseudo-traces*

From the classical theory of heat expansions [9,11], [14, (3.18)], for a positive elliptic operator Q of positive integer order and a PDO A acting on sections of a vector bundle over a closed manifold M, the map $\varepsilon \to \mathrm{tr}(A\,\mathrm{e}^{-\varepsilon Q})$ lies in the class $\mathcal{F}_q$ of Section 4, where $q = \mathrm{ord}(Q)$. More precisely, there exist $N = N(\dim M) \in \mathbf{N}^+$, $a = a(\mathrm{ord}(A)) \in \mathbf{R}$ and $\alpha_j(Q, A), \beta_j(Q, A), \gamma_j(Q, A) \in \mathbf{C}$ such that

$$\begin{aligned}\mathrm{tr}(A\,\mathrm{e}^{-\varepsilon Q}) \sim{} & \sum_{j=0}^{\infty} \alpha_j(Q, A) \varepsilon^{(j-a-N)/q} \\ & + \sum_{j=0,((j-a-N)/q)\in\mathbf{Z}}^{\infty} \beta_j(Q, A) \varepsilon^{(j-a-N)/q} \log\varepsilon + \sum_{j=0}^{\infty} \gamma_j(Q, A) \varepsilon^j\end{aligned} \tag{5.3}$$

as $\varepsilon \to 0$. As before, constant terms can arise as both γ_0 and α_{a+N} if $\mathrm{ord}(A) \in \mathbf{Z}$. We define the *Q-renormalized trace* of A as the μ-renormalized limit of the map $\varepsilon \mapsto \mathrm{tr}(A\,\mathrm{e}^{-\varepsilon Q})$, as in (4.3)

$$\mathrm{tr}^{Q,\mu}(A) \equiv \mathrm{Lim}_{\varepsilon\to 0}^{\mu}\, \mathrm{tr}_\varepsilon^Q(Q) = \alpha_{a+N}(Q, A) + \gamma_0(Q, A) - \mu \cdot \beta_{a+N}(Q, A). \tag{5.4}$$

If we set $\mu = 0$, respectively $\mu = \gamma$, the Euler constant, we get a heat kernel renormalized trace, respectively, a zeta function renormalized trace, and these two are related via a Mellin transform. In the following, we will usually consider the case $\mu = 0$, and write tr^Q for $\mathrm{tr}^{Q,0}$. The results can easily be extended to the general case of $\mathrm{tr}^{Q,\mu}$.

5.3. Renormalized supertraces

(5.4) extends to Q-renormalized supertraces in the obvious way for $Q = Q^+ \oplus Q^-$:

$$\mathrm{str}^{Q,\mu}(A) = \mathrm{tr}^{Q^+,\mu}(A^+) - \mathrm{tr}^{Q^-,\mu}(A^-),$$

for A as in (4.2) with $A^\pm$ PDOs. Renormalized pseudo-traces/supertraces appear in the geometry of determinant bundles [4], where the connection on the determinant bundle can be written as

$$\begin{aligned}(\mathrm{Det}\, L_b^+)^{-1}\nabla^{\mathrm{Det},\mu}\,\mathrm{Det}\, L_b^+ &\equiv \mathrm{tr}^{Q^+,\mu}((L_b^+)^{-1}\nabla^{\mathrm{Hom}(\mathcal{E})}L_b^+)\\ &= \tfrac{1}{2}(\mathrm{dlog}\,\mathrm{det}_\mu Q^+ + \mathrm{str}^{Q,\mu}((L_b)^{-1}\nabla^{\mathrm{Hom}(\mathcal{E})}L_b)).\end{aligned} \tag{5.5}$$

They also have been used (i) to define minimality of infinite dimensional submanifolds of manifolds of connections and metrics [2,15] and (ii) in relation to determinants of elliptic operators [12], for a special class of operators on which they are actually traces.

These renormalized traces are related to Wodzicki residues, as we briefly recall; see [12,13,16] for more details. Let $(\mathcal{E}, Q)$ in $\mathcal{CH}$ be a weighted vector bundle with fibers modeled on $H^s(M, E)$, and let A be a section of $CL(\mathcal{E})$. Since Q is positive elliptic with strictly positive order for any $z \in \mathbf{C}$ with $\mathrm{Re}(z) > \dim M/\mathrm{ord}(Q)$, the operator $(Q+P_Q)^{-z}$ is trace-class on $L^2(M, E)$ in any local chart. Here P_Q is the orthogonal projection onto the kernel of Q. Similarly, for $\mathrm{Re}(z) > (\dim M + \mathrm{ord}(A))/\mathrm{ord}(Q)$, $A(Q+P_Q)^{-z}$ is trace-class. For such z, we may define

$$\widetilde{\mathrm{str}}_z^Q(A) \equiv \mathrm{str}(A(Q + P_Q)^{-z}).$$

Because $A^\pm$, $Q^\pm$, $P_{Q^\pm}$ are classical PDOs, it is standard that $\widetilde{\mathrm{str}}_z^Q(A)$ has a meromorphic continuation to $\mathbf{C}$ with at most simple poles. By the Mellin transform, we have

$$\beta_{a+n}(Q, A) = \mathrm{res}_{z=0}(\widetilde{\mathrm{str}}_z^Q(A))$$

in the notation of (5.3); in particular, $\beta_{a+n}(A) = \beta_{a+n}(Q, A)$ is independent of Q. It follows via a Mellin transform that

$$\mathrm{str}^{Q,\mu}(A) = \lim_{z\to 0}(\widetilde{\mathrm{str}}_z^Q(A) - z^{-1}\,\mathrm{res}_{z=0}(\widetilde{\mathrm{str}}_z^Q(A)) + (\gamma - \mu)\,\mathrm{res}_{z=0}(\widetilde{\mathrm{str}}_z^Q(A)).$$

Renormalized pseudo-traces thus arise as the finite part of a divergent expression. The infinite part is built from the Wodzicki residue [21] $\mathrm{res}(A)$:

$$\mathrm{res}(A) \equiv (\mathrm{ord}(Q))\,\mathrm{res}_{z=0}(\widetilde{\mathrm{str}}_z^Q(A)), \tag{5.6}$$

which defines a trace on the algebra of PDOs [11,21]. In summary

$$\mathrm{str}^{Q,\mu}(A) = \lim_{z\to 0}\left(\widetilde{\mathrm{str}}_z^Q(A) - \frac{1}{z\,\mathrm{ord}(Q)}\mathrm{res}(A)\right) + \frac{\gamma-\mu}{\mathrm{ord}(Q)}\mathrm{res}(A).$$

We can now define Q-weighted first Chern forms on a weighted vector bundle.

Definition. Let $(\mathcal{E}, Q)$ be a weighted Hermitian (super) vector bundle over B with connection $\nabla^{\mathcal{E}}$ and curvature $\Omega^{\mathcal{E}}$. Assume that for any $X, Y \in \Gamma(TB)$, $\Omega^{\mathcal{E}}(X, Y) \in \Gamma(CL(\mathcal{E}))$. Define

(i) the one-parameter family of *Q-weighted first Chern forms by*

$$r_1^{Q,\varepsilon}(X, Y) \equiv \mathrm{str}_\varepsilon^Q(\Omega^{\mathcal{E}}(X, Y)), \quad \varepsilon > 0, \tag{5.7}$$

(ii) the one-parameter family of *Q-renormalized first Chern forms*

$$R_1^{Q,\mu}(X, Y) \equiv \mathrm{str}^{Q,\mu}(\Omega^{\mathcal{E}}(X, Y)), \quad \mu \in \mathbf{R}. \tag{5.8}$$

6. The curvature on the associated determinant bundle in finite dimensions

Let $\mathcal{E}$ be a finite dimensional bundle with connection $\nabla^{\mathcal{E}}$, and let α be a $\mathrm{Hom}(\mathcal{E}, \mathcal{E})$-valued form. Writing $\nabla^{\mathcal{E}} = \mathrm{d} + \theta$ in a local trivialization, we have

$$d\,\mathrm{tr}(\alpha) = \mathrm{tr}([d, \alpha]) = \mathrm{tr}([d, \alpha]) + \mathrm{tr}([\theta, \alpha]) = \mathrm{tr}([\nabla^{\mathcal{E}}, \alpha]), \tag{6.1}$$

since the trace term $\mathrm{tr}([\theta, \alpha])$ vanishes. The final expression is of course independent of the choice of local trivialization. Thus the trace of a covariantly constant form is closed. In particular, since the curvature $\Omega^{\mathcal{E}}$ is covariantly constant by the Bianchi identity, the first Chern form $r_1^{\mathcal{E}} \equiv \mathrm{tr}(\Omega^{\mathcal{E}})$ is also closed. This form is a representative of the first Chern class in de Rham cohomology.

This generalizes to supertraces on superbundles

$$\mathrm{d}\,\mathrm{str}(\alpha) = \mathrm{str}([\nabla^{\mathcal{E}}, \alpha]), \tag{6.2}$$

where $[\cdot, \cdot]$ is now a supercommutator and $\nabla^{\mathcal{E}}$ a superconnection on the superbundle $\mathcal{E}$. The first Chern form $r_1^{\mathcal{E}} \equiv \mathrm{str}(\Omega^{\mathcal{E}})$ is therefore also closed.

We recall the relation between the first Chern form of a superbundle and the curvature of the associated determinant bundle. Let $\mathcal{E}^\pm$ be Hermitian vector bundles with connections $\nabla^{\mathcal{E}\pm}$ over a manifold B. $\nabla^{\mathcal{E}\pm}$ induce a connection $\nabla^{\mathcal{E}}$ on $\mathcal{E} = \mathcal{E}^+ \oplus \mathcal{E}^-$. The bundle $\mathrm{Hom}(\mathcal{E}^+, \mathcal{E}^-) \simeq (\mathcal{E}^+)^* \otimes \mathcal{E}^-$ has the natural connection $\nabla^{\mathrm{Hom}(\mathcal{E})} \equiv (\nabla^{\mathcal{E}^+})^* \otimes 1 + 1 \otimes \nabla^{\mathcal{E}^-}$, given by $\nabla^{\mathrm{Hom}(\mathcal{E})} L^+ = [\nabla^{\mathcal{E}}, L^+]$ for $L^+ \in \Gamma(\mathrm{Hom}(\mathcal{E}^+, \mathcal{E}^-))$ (cf. Appendix A). Assuming for convenience that $\mathcal{E}^\pm$ have the same rank, the determinant bundle $\mathrm{Det}(\mathcal{E}) \equiv (\Lambda^{\mathrm{top}}\mathcal{E}^+)^* \otimes \Lambda^{\mathrm{top}}\mathcal{E}^-$ has the Hermitian metric

$$\|\mathrm{Det}\, L^+\| \equiv \sqrt{\det((L^+)^* L^+)}$$

for $L^+ \in \Gamma(\mathrm{Hom}(\mathcal{E}^+, \mathcal{E}^-))$ and $\mathrm{Det}\, L^+$ the corresponding section of $\mathrm{Det}(\mathcal{E}^+, \mathcal{E}^-)$. $\nabla^{\mathcal{E}}$ induces a connection $\nabla^{\mathrm{Det}\,\mathcal{E}}$ on $\mathrm{Det}(\mathcal{E})$ compatible with this metric, defined at points where L^+ is injective by

$$\begin{aligned}(\mathrm{Det}\, L^+)^{-1} \nabla^{\mathrm{Det}\,\mathcal{E}}\, \mathrm{Det}\, L^+ &\equiv \mathrm{tr}((L^+)^{-1}[\nabla^{\mathcal{E}}, L^+]) \\ &= \tfrac{1}{2}(\mathrm{dlog}\det Q^+ + \mathrm{str}(L^{-1}[\nabla^{\mathcal{E}}, L])),\end{aligned} \tag{6.3}$$

where $L = L^+ \oplus (L^+)^*$, $Q^+ = (L^+)^* L^+$ (cf. (5.3) and (5.5)). The following lemma is well known.

Lemma 1. *The curvature* $\Omega^{\mathrm{Det}\,\mathcal{E}}$ *of the connection* $\nabla^{\mathrm{Det}\,\mathcal{E}}$ *on the determinant bundle* $\mathrm{Det}(\mathcal{E})$ *associated to the connection* $\nabla^{\mathcal{E}}$ *on the superbundle* $\mathcal{E}=\mathcal{E}^+\oplus\mathcal{E}^-$ *satisfies*

$$\Omega^{\mathrm{Det}\,\mathcal{E}} = -\mathrm{str}(\Omega^{\mathcal{E}}) = \mathrm{ch}(\nabla^{\mathcal{E}})_{[2]}, \tag{6.4}$$

where $\Omega^{\mathcal{E}}$ *is the curvature of* $\nabla^{\mathcal{E}}$, *and* $\mathrm{ch}(\nabla^{\mathcal{E}})_{[2]}$ *is the degree two component of the Chern character* $\mathrm{str}(\exp[-\Omega^{\mathcal{E}}])$ *of the connection, i.e. the curvature of the determinant line bundle is minus the first Chern form of the superbundle.*

Proof. For later purposes, we give a basis free proof. Pick $M, N, \in T_bB$, where L^+ is injective at b. Extend L^+ near b so that $[\nabla^{\mathcal{E}}, L^+]_b = 0$. By (6.3), we have

$$\Omega^{\mathrm{Det}\,\mathcal{E}}(M,N) = \tfrac{1}{2}\,\mathrm{d}(\mathrm{str}(L^{-1}[\nabla^{\mathcal{E}}, L]))(M,N) = \tfrac{1}{2}\,\mathrm{str}([\nabla^{\mathcal{E}}, L^{-1}[\nabla^{\mathcal{E}}, L]])(M,N).$$

Using the Cartan formula $\mathrm{d}\alpha(M,N) = M(\alpha(N)) - N(\alpha(M)) - \alpha([M,N])$, we get

$$\begin{aligned}\Omega^{\mathrm{Det}\,\mathcal{E}}(M,N) &= \tfrac{1}{2}\mathrm{str}([L^{-1}[\nabla^{\mathcal{E}}_M, L], L^{-1}[\nabla^{\mathcal{E}}_N, L]]) + \mathrm{str}(L^{-1}[\Omega^{\mathcal{E}}, L])(M,N)) \\ &= \tfrac{1}{2}\,\mathrm{str}(L^{-1}[\Omega^{\mathcal{E}}, L])(M,N) = -\mathrm{str}(\Omega^{\mathcal{E}})(M,N),\end{aligned} \tag{6.5}$$

where we have used $\mathrm{str}\,(A^{-1}[B,A]) = -2\mathrm{str}\,(B)$ for A odd, B even. The second equality in (6.4) is standard. □

7. The curvature on the determinant bundle in infinite dimensions

The main goal of this paper is to see how (6.4) extends to the infinite dimensional setting. More precisely, the Quillen–Bismut–Freed theory of determinant bundles constructs a determinant bundle with connections (5.2) and (5.5), for certain half-weighted superbundles, with the curvature of (5.5) computed in [4]. Via weighted traces, we have constructed weighted and renormalized first Chern forms of such superbundles, and it is natural to ask if (6.4) continues to hold.

The proof of (6.4) uses the facts $\mathrm{tr}([A,B]) = 0$ and $\mathrm{d}\,\mathrm{str} = \mathrm{str}([\nabla^{\mathcal{E}}, \cdot])$, both of which fail for weighted traces. Thus we cannot expect (6.4) to hold in infinite dimensions. Indeed we will show by two methods that (6.4) holds up to an obstruction. The two methods lead to different expressions for these obstructions which seem difficult to identify directly.

The first zeta function regularization approach uses weighted traces to express the supertrace of a commutator and the obstruction to $\mathrm{d}\,\mathrm{str} = \mathrm{str}([\nabla, \cdot])$ in terms of Wodzicki residues. The appearance of Wodzicki residues is natural, since they are defined via zeta function regularization. The second heat kernel regularization approach uses a one-parameter family of superconnections introduced by Bismut [3] to avoid weighted traces, and closely follows the methods used in [4,5] to compute the curvature on the determinant bundle for families of Dirac operators.

7.1. First approach using weighted traces

Let $\mathcal{E} = \mathcal{E}^+ \oplus \mathcal{E}^-$ in $\mathcal{CE}$ be a superbundle with connection $\nabla^{\mathcal{E}}$ with even transition maps acting on a model space $H^s(M, E)$, $s > (\dim M)/2$. Let Q be a weight on $\mathcal{E}$. The following lemma expresses the obstruction to $\mathrm{d}\,\mathrm{str} = \mathrm{str}([\nabla, \cdot])$ as a Wodzicki residue.

Lemma 2 ([6]). *Let $(\mathcal{E}, Q)$ be a weighted vector bundle with connection ∇ over a manifold B, and let α, β, be sections of the bundle $CL(\mathcal{E})$ based on B. For $\mu \in \mathbf{R}$,*

(1)

$$\mathrm{str}^{Q,\mu}[\alpha, \beta] = -\frac{1}{\mathrm{ord}(Q)}\,\mathrm{res}([\log Q, \alpha]\beta). \tag{7.1}$$

(2) *If $[\nabla, \log Q]$ and $[\nabla, \alpha]$ are $CL(\mathcal{E})$-valued one-forms, then*

$$\mathrm{d}(\mathrm{str}^{Q,\mu}(\alpha)) = \mathrm{str}^{Q,\mu}([\nabla, \alpha]) - \frac{1}{\mathrm{ord}(Q)}\,\mathrm{res}(\alpha \cdot [\nabla, \log Q]). \tag{7.2}$$

For completeness, we outline the proof of (7.2) for traces, which easily extends to supertraces, and refer the reader to [6] for (7.1). As before, tr^Q denotes the renormalized trace $\mathrm{tr}^{Q,\mu}$ at $\mu = 0$; the results extend to $\mu \neq 0$.

One first shows that for one-parameter families of operators $A_t \in CL(M, E)$, $Q_t \in \mathrm{Ell}^+_{\mathrm{ord}>0}(M, E)$ of constant order, we have

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (\mathrm{tr}^{Q_t}(A_t)) = \mathrm{tr}^{Q_0}\left(\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} A_t\right) - \frac{1}{\mathrm{ord}(Q)_0}\,\mathrm{res}\left(A_0 \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \log Q_t\right).$$

This uses the fundamental property of the canonical trace of Kontsevich–Vishik [13]. Similarly, in a fixed local trivialization of $\mathcal{E}$, we have

$$\mathrm{d}\,\mathrm{tr}^Q(\alpha) = \mathrm{tr}^Q(\mathrm{d}\alpha) - \frac{1}{\mathrm{ord}(Q)}\,\mathrm{res}(\alpha\,\mathrm{dlog}\,Q). \tag{7.3}$$

Let $\nabla = \mathrm{d} + \theta$ in the local trivialization. Since $[\nabla, \alpha] = \mathrm{d}\alpha + [\theta, \alpha] \in \Gamma(CL(\mathcal{E}))$, and since $\mathrm{d}\alpha$, the differential of a PDO, also lies in $\Gamma(CL(\mathcal{E}))$, it follows that $[\theta, \alpha]$ lies in $CL(M, E)$ pointwise. Using again the fundamental property of the canonical trace, one shows

$$\mathrm{tr}^Q[\theta, \alpha] = -\frac{1}{\mathrm{ord}(Q)}\,\mathrm{res}([\log Q, \theta]\alpha). \tag{7.4}$$

Combining (7.3) and (7.4) gives

$$\begin{aligned}
\mathrm{d}\,\mathrm{tr}^Q(\alpha) &= \mathrm{tr}^Q(\mathrm{d}\alpha) - \frac{1}{\mathrm{ord}(Q)}\,\mathrm{res}(\alpha\,\mathrm{dlog}\,Q)\\
&= \mathrm{tr}^Q([\nabla, \alpha]) - \mathrm{tr}^Q([\theta, \alpha]) - \frac{1}{\mathrm{ord}(Q)}\,\mathrm{res}(\alpha\,\mathrm{dlog}\,Q)\\
&= \mathrm{tr}^Q([\nabla, \alpha]) + \frac{1}{\mathrm{ord}(Q)}\,\mathrm{res}([\log Q, \theta]\alpha) - \frac{1}{\mathrm{ord}(Q)}\,\mathrm{res}(\alpha\,\mathrm{dlog}\,Q)\\
&= \mathrm{tr}^Q([\nabla, \alpha]) - \frac{1}{\mathrm{ord}(Q)}\,\mathrm{res}(\alpha[\nabla, \log Q]).
\end{aligned}$$

□

The residue term in (7.2) is the source of the infinite dimensional obstruction to identifying the first Chern form of a superbundle with (minus) the curvature of the determinant bundle.

Theorem 3. *Let* $(\mathcal{E} = \mathcal{E}^+ \oplus \mathcal{E}^-, L = L^+ \oplus L^-)$ *be a half-weighted super vector bundle with connection* $\nabla^{\mathcal{E}}$ *over a manifold B. The curvature* $\Omega^{\mathrm{Det},\mu}$ *of the associated determinant bundle differs from the Q-weighted first Chern form* $R_1^{Q,\mu}$ *of (5.8) on the weighted superbundle* $(\mathcal{E}, Q = L^2)$ *by a Wodzicki residue. More precisely, for* $M, N \in T_bB$ *we have*

$$\Omega^{\mathrm{Det},\mu}(M,N) = -R_1^{Q,\mu}(M,N) + \mathcal{R}^{Q,\nabla^{\mathcal{E}}}(M,N),$$

with

$$\begin{aligned}2\mathcal{R}^{Q,\nabla^{\mathcal{E}}}(M,N) = \frac{1}{\mathrm{ord}(Q)}&\,\mathrm{res}([\log Q, L^{-1}[\nabla_M^{\mathcal{E}}, L]]L^{-1}[\nabla_N^{\mathcal{E}}, L] - L^{-1}[\nabla_N^{\mathcal{E}}, L]\\ &\cdot[\nabla_M^{\mathcal{E}}, \log Q] + L^{-1}[\nabla_M^{\mathcal{E}}, L]\cdot[\nabla_N^{\mathcal{E}}, \log Q]).\end{aligned}$$

Proof. We follow the proof of Lemma 1, replacing traces by renormalized supertraces and keeping track of obstructions due to (7.1) and (7.2) via Wodzicki residues. Dropping μ, we obtain

$$\begin{aligned}2\Omega^{\mathrm{Det}}(M,N) &= \mathrm{d}(\mathrm{str}^Q(L^{-1}[\nabla^{\mathcal{E}}, L]))(M,N)\\ &= -\mathrm{str}^Q([L^{-1}[\nabla_M^{\mathcal{E}}, L], L^{-1}[\nabla_N^{\mathcal{E}}, L]]) + \mathrm{str}^Q(L^{-1}[\Omega^{\mathcal{E}}(M,N), L])\\ &\quad -\frac{1}{\mathrm{ord}(Q)}\mathrm{res}(L^{-1}[\nabla_N^{\mathcal{E}}, L]\cdot[\nabla_M^{\mathcal{E}}, \log Q])\\ &\quad +\frac{1}{\mathrm{ord}(Q)}\mathrm{res}(L^{-1}[\nabla_M^{\mathcal{E}}, L]\cdot[\nabla_N^{\mathcal{E}}, \log Q]),\end{aligned}$$

using (7.2) and calculating as in (6.5). Thus

$$\begin{aligned}2\Omega^{\mathrm{Det}}(M,N) &= -2\mathrm{str}^Q(\Omega^{\mathcal{E}}(M,N))\\ &\quad +\frac{1}{\mathrm{ord}(Q)}\mathrm{res}([\log Q, L^{-1}[\nabla_M^{\mathcal{E}}, L]]L^{-1}[\nabla_N^{\mathcal{E}}, L])\\ &\quad -\frac{1}{\mathrm{ord}(Q)}\mathrm{res}(L^{-1}[\nabla_N^{\mathcal{E}}, L]\cdot[\nabla_M^{\mathcal{E}}, \log Q])\\ &\quad +\frac{1}{\mathrm{ord}(Q)}\mathrm{res}(L^{-1}[\nabla_M^{\mathcal{E}}, L]\cdot[\nabla_N^{\mathcal{E}}, \log Q]),\end{aligned}$$

using (7.1). □

7.2. The heat kernel approach

Here we deform the weight $Q = Q_0 \equiv L^2$ to a one-parameter family $Q_0 + Q_{1,\varepsilon}, \varepsilon > 0$ via a deformation of the superconnection $\nabla^{\mathcal{E}}$ into a family ∇_ε^L of Bismut superconnections. We need a preliminary formula.

7.3. Volterra series

Let $Q = Q_0 + Q_1$, where Q_0 is a positive elliptic operator of strictly positive order, and Q_1 is a PDO of order strictly less than that of Q_0. We have

$$\mathrm{e}^{-\varepsilon(Q_0+Q_1)} = \sum_{k=0}^{\infty}(-\varepsilon)^k \int_{\Delta^k} \mathrm{e}^{-\sigma_0\varepsilon Q_0} Q_1\, \mathrm{e}^{-\sigma_1\varepsilon Q_0} Q_1, \dots, Q_1\, \mathrm{e}^{-\sigma_k\varepsilon Q_0}\, \mathrm{d}\sigma_0\, \mathrm{d}\sigma_1, \dots, \mathrm{d}\sigma_k,$$

where $\Delta^k = \{\sigma_0, \dots, \sigma_k > 0 : \sum_{i=0}^{k} \sigma_i = 1\}$ [5, (2.5)]. We can avoid convergence issues, since we will only be using a finite number of terms. In analogy with the notation in [10], we set

$$\langle A_0, A_1, \dots, A_k\rangle_{\varepsilon,k,Q_0} \equiv \int_{\Delta^k} \mathrm{str}(A_0\, \mathrm{e}^{-\sigma_0\varepsilon Q_0} A_1\, \mathrm{e}^{-\sigma_1\varepsilon Q_0} A_2 \cdots A_k\, \mathrm{e}^{-\sigma_k\varepsilon Q_0})\, \mathrm{d}\sigma_0\, \mathrm{d}\sigma_1 \cdots \mathrm{d}\sigma_k,$$

for PDOs $A_0, \dots, A_k$ acting on sections of the model bundle E of $\mathcal{E}$. The supertrace is clearly finite for $\varepsilon > 0$. The Volterra formula implies

$$\mathrm{str}(A\, \mathrm{e}^{-\varepsilon(Q_0+Q_1)}) = \sum_{k=0}^{\infty}(-\varepsilon)^k \langle A, Q_1, \dots, Q_1\rangle_{\varepsilon,k,Q_0},$$

for any PDO A.

7.4. Bismut superconnections

Starting from a half-weighted superbundle $(\mathcal{E} = \mathcal{E}^+ \oplus \mathcal{E}^-, L)$ with a metric superconnection $\nabla^{\mathcal{E}} = \nabla^+ \oplus \nabla^-$, we form the one-parameter family of superconnections

$$\nabla_\varepsilon^L \equiv \nabla^{\mathcal{E}} + \sqrt{\varepsilon} L,$$

for $\varepsilon > 0$ [3]. For any one-parameter family of superconnections A_t, we have the important *transgression formula*: for an analytic function f,

$$\frac{\mathrm{d}}{\mathrm{d}t}\, \mathrm{str}(f(A_t^2)) = \mathrm{d}\left(\mathrm{str}\left(\frac{\mathrm{d}}{\mathrm{d}t} A_t\, f'(A_t^2)\right)\right). \tag{7.5}$$

The derivation of this formula in [5, Proposition 1.41] essentially relies on the fact that $\mathrm{d}(\mathrm{str}(\alpha)) = \mathrm{str}[A, \alpha]$ for $\mathrm{Hom}(\mathcal{E}, \mathcal{E})$-valued one-forms α. Thus in the proof below, we avoid the obstructions to (6.1) and (6.2), for the weighted supertraces str^Q.

Proposition 4. *Let $(\mathcal{E}, L)$ be a half-weighted superbundle with connection $\nabla^{\mathcal{E}}$ over a manifold B. For $\varepsilon > 0$ we have*

$$\Omega^{\mathrm{Det}\,\mathcal{E},\varepsilon} = \mathrm{ch}(\nabla_\varepsilon^L)_{[2]} = -r_1^{Q,\varepsilon} + \varepsilon\langle I, [\nabla^{\mathcal{E}}, L], [\nabla^{\mathcal{E}}, L]\rangle_{\varepsilon,2,Q_0},$$

where $Q_0 = L^2$, $r_1^{Q,\varepsilon} \equiv \mathrm{str}_\varepsilon^{Q_0}(\Omega^{\mathcal{E}})$ is the weighted first Chern form of $\nabla^{\mathcal{E}}$ as in (5.7) $\Omega^{\mathrm{Det}\,\mathcal{E},\varepsilon}$ is the curvature of $\nabla^{\mathrm{Det}\,\mathcal{E},\varepsilon}$ as in (4.1), and $\mathrm{ch}(\nabla_\varepsilon^L)_{[2]} \equiv \mathrm{str}(\exp(-(\nabla_\varepsilon^L)^2))_{[2]}$ is the degree two component of the Chern character of ∇_ε^L.

Proof. We first compute the degree k piece of the Chern character of ∇_ε^L in two ways. On the one hand, since

$$(\nabla_\varepsilon^L)^2 = \varepsilon Q_0 + \sqrt{\varepsilon}[\nabla^{\mathcal{E}}, L] + (\nabla^{\mathcal{E}})^2 = \varepsilon(Q_0 + Q_{1,\varepsilon}),$$

with $Q_0 \equiv L^2$, $Q_{1,\varepsilon} \equiv (1/\sqrt{\varepsilon})[\nabla^{\mathcal{E}}, L] + (1/\varepsilon)(\nabla^{\mathcal{E}})^2$, by the Volterra formula we have

$$\mathrm{ch}(\nabla_\varepsilon^L)_{[k]} = \sum_{j=0}^{\infty}(-\varepsilon)^j[\langle I, Q_{1,\varepsilon}, \dots, Q_{1,\varepsilon}\rangle_{\varepsilon,j,Q_0}]_{-k}. \tag{7.6}$$

As promised, we need only consider a finite number of terms in this sum.

On the other hand, we have

$$\begin{aligned}\mathrm{ch}(\nabla_\varepsilon^L)_{[k]} &= \mathrm{str}[\exp[-(\nabla_\varepsilon^L)^2]]_{[k]} = \mathrm{str}\int_\varepsilon^\infty \left[\frac{\mathrm{d}}{\mathrm{d}t}(\exp[-(\nabla_t^L)^2)\right]_{[k]}\mathrm{d}t \\ &= -\int_\varepsilon^\infty \frac{\mathrm{d}}{\mathrm{d}t}[\mathrm{str}(\exp[-(\nabla_t^L)^2])]_{[k]}\,\mathrm{d}t.\end{aligned}$$

Applying the transgression formula (7.5) to $f(x) = \mathrm{e}^{-x}$, we get

$$\begin{aligned}\mathrm{ch}(\nabla_\varepsilon^L)_{[k]} &= \int_\varepsilon^\infty \mathrm{d}\left[\mathrm{str}\left(\left(\frac{\mathrm{d}}{\mathrm{d}t}\nabla_t^L\right)\exp[-(\nabla_t^L)^2]\right)\right]_{[k-1]}\mathrm{d}t \\ &= \frac{1}{2}\left(\mathrm{d}\left[\int_\varepsilon^\infty \mathrm{str}\left(\frac{L}{\sqrt{t}}\exp[-(\nabla_t^L)^2]\right)\mathrm{d}t\right]_{[k-1]}\right) \\ &= \frac{1}{2}\,\mathrm{d}\int_\varepsilon^\infty \sum_{j=0}^{\infty}\frac{(-t)^j}{\sqrt{t}}[\langle L, Q_{1,t}, \dots, Q_{1,t}\rangle_{t,j,Q_0}]_{[k-1]}.\end{aligned} \tag{7.7}$$

Combining (7.6) and (7.7) yields

$$\begin{aligned}&\sum_{j=0}^{\infty}(-\varepsilon)^j[\langle I, Q_{1,\varepsilon}, \dots, Q_{1,\varepsilon}\rangle_{\varepsilon,j,Q_0}]_{[k]} \\ &\quad = \frac{1}{2}\,\mathrm{d}\int_\varepsilon^\infty \sum_{j=0}^{\infty}\frac{(-t)^j}{\sqrt{t}}[\langle L, Q_{1,t}, \dots, Q_{1,t}\rangle_{t,j,Q_0}]_{[k-1]}.\end{aligned} \tag{7.8}$$

Since $Q_0 = L^2$, we have

$$\begin{aligned}\Omega^{\mathrm{Det}\,\mathcal{E},\varepsilon} &= \frac{1}{2}\,\mathrm{d}(\mathrm{str}(L^{-1}[\nabla^{\mathcal{E}}, L]\,\mathrm{e}^{-\varepsilon Q_0})) = -\frac{1}{2}\,\mathrm{d}\int_\varepsilon^\infty \frac{\mathrm{d}}{\mathrm{d}t}\,\mathrm{str}(L^{-1}[\nabla^{\mathcal{E}}, L]\,\mathrm{e}^{-tQ_0})\,\mathrm{d}t \\ &= \frac{1}{2}\,\mathrm{d}\int_\varepsilon^\infty \mathrm{str}(L[\nabla^{\mathcal{E}}, L]\,\mathrm{e}^{-tQ_0})\,\mathrm{d}t = \frac{1}{2}\,\mathrm{d}\int_\varepsilon^\infty \langle L, [\nabla^{\mathcal{E}}, L]\rangle_{t,1,Q_0}\,\mathrm{d}t \\ &= \frac{1}{2}\,\mathrm{d}\left[\int_\varepsilon^\infty \sqrt{t}\langle L, Q_{1,t}\rangle_{t,1,Q_0}\,\mathrm{d}t\right]_{[1]} \\ &= -\frac{1}{2}\,\mathrm{d}\int_\varepsilon^\infty \left[\sum_{j=0}^{\infty}\frac{(-t)^j}{\sqrt{t}}\langle L, Q_{1,t}, \dots, Q_{1,t}\rangle_{t,j,Q_0}\right]_{[1]}\mathrm{d}t.\end{aligned} \tag{7.9}$$

We used $[L, Q_0] = 0$ in the second line, and the last equality follows since the only term of degree one in the infinite sum is the integrand in the next to last integral. By (7.7) and (7.9), we see that $\Omega^{\mathrm{Det}\,\mathcal{E},\varepsilon} = -\mathrm{ch}(\nabla^L_\varepsilon)_{[2]}$.

Finally, by (7.8) and (7.9), we get

$$\begin{aligned}\Omega^{\mathrm{Det}\,\mathcal{E},\varepsilon} &= -\langle I, (\nabla^{\mathcal{E}})^2\rangle_{\varepsilon,1,Q_0} + \varepsilon\langle I, [\nabla^{\mathcal{E}}, L], [\nabla^{\mathcal{E}}, L]\rangle_{\varepsilon,2,Q_0} \\ &= -\mathrm{str}(\Omega^{\mathcal{E}}\,\mathrm{e}^{-\varepsilon Q_0}) + \varepsilon\langle I, [\nabla^{\mathcal{E}}, L], [\nabla^{\mathcal{E}}, L]\rangle_{\varepsilon,2,Q_0},\end{aligned}$$

which finishes the proof. □

Remark. In fact, (7.8) vanishes for k odd, since the integrand is the supertrace of an odd operator, and hence vanishes.

By taking the renormalized limit in Proposition 4, we obtain the following theorem.

Theorem 5. *For any $\mu \in \mathbf{R}$, the renormalized first Chern form $R_1^{Q,\mu}$ defined in (5.8) and the curvature $\Omega^{\mathrm{Det},\mu}$ of the determinant line bundle are related by*

$$\Omega^{\mathrm{Det},\mu} = \mathrm{Lim}^{\mu}_{\varepsilon\to 0}\,\mathrm{ch}(\nabla^L_\varepsilon)_{[2]} = -R_1^{Q,\mu} + \mathrm{Lim}^{\mu}(\varepsilon\langle I, [\nabla^{\mathcal{E}}, L], [\nabla^{\mathcal{E}}, L]\rangle_{\varepsilon,2,Q_0}).$$

Our approach differs somewhat from [4], as Bismut and Freed calculate $\Omega^{\mathrm{Det},0}$ as $\mathrm{Lim}^0_{\varepsilon\to 0}\,\mathrm{ch}(\tilde{\nabla}^L_\varepsilon)_{[2]}$, where $\tilde{\nabla}^L_\varepsilon = \nabla^{\mathcal{E}} + \varepsilon^{1/2}L + \varepsilon^{-1/2}A_2$ is the Bismut superconnection (for an explicit term A_2 of degree two). The proof of Proposition 4 applies to $\tilde{\nabla}^L_\varepsilon$; starting with (7.7), we have additional terms involving A_2, which do not contribute to $\Omega^{\mathrm{Det},\mu}$, the $k = 2$ term in (7.7). Therefore, the degree-two pieces of the Chern characters of the superconnections ∇^L_ε, $\tilde{\nabla}^L_\varepsilon$ both compute the curvature of the determinant line bundle, although the higher degree pieces differ. In the next paragraph, we will relate these expressions.

7.5. Bismut–Freed connections

For the connection on the infinite dimensional bundle $\mathcal{E} = \mathcal{E}^+ \oplus \mathcal{E}^-$ considered in [4], we can say more about the renormalized first Chern form. As in (3.4), we consider a fibration $\pi : Z \to B$ of manifolds, with fiber an even dimensional spin manifold M_b, $b \in B$, and with finite rank Hermitian bundles $E^\pm$ with unitary connections over Z. The Levi-Civita connection ∇^{LC} for a given metric on Z and the associated orthogonal horizontal splitting $T_xZ = T_bM_b \oplus H_x$, for $x \in \pi^{-1}(b)$, induces a connection ∇^F on F, the tangent bundle along the fibers of π by

$$\nabla^F = P^{TM}\nabla^{\mathrm{LG}}, \tag{7.10}$$

where P^{TM} is the orthogonal projection to F. We lift ∇^F to a connection on the spinor bundle $S = S^+ \oplus S^-$ associated to F. For an auxiliary bundle with connection W on Z, we set $\mathcal{E} = \mathcal{E}^+ \oplus \mathcal{E}^-$, a bundle over B with fiber $H^s(M_b, (S^\pm \otimes W)|_{M_b})$ for $\mathcal{E}^\pm$. The induced connection ∇^E on $E = S \otimes W$ in turn induces a connection $\tilde{\nabla}$ on $\mathcal{E}$ given by

$$(\tilde{\nabla}_Y h)(b)(x) = (\nabla^E_{\tilde{Y}} h)(x), \tag{7.11}$$

where $\tilde{Y}$ is the horizontal lift of $Y \in TB$ to H_x. This connection need not be unitary, but it is shown in [4] that adding the divergence of the volume form at x in base directions to the right-hand side of (7.11) produces a unitary superconnection $\tilde{\nabla}^u$, the Bismut–Freed connection on $\mathcal{E}$.

We claim that the curvature form Ω^u for $\tilde{\nabla}^u$ is an endomorphism in the fibers:

$$\Omega_b^u \in \Lambda^2(\mathrm{Hom}(S \otimes W|_{M_b}, S \otimes W|_{M_b})).$$

First, the local nature of the Bismut–Freed connection show that if $\psi \in \Gamma(\mathcal{E})$ has support in $U \times V$, where U is a open set in B and V is an open set containing x in $\pi^{-1}(U)$, then $\tilde{\nabla}^u \psi$ also has support in $U \times V$. This implies the same result for $\Omega^u(X, Y)$, a combination of first and second covariant derivatives of $\tilde{\nabla}^u$. Since Ω^u is tensorial, after multiplying ψ by bump functions with decreasing support in base directions, we can shrink $U \times V$ to the point b. In other words, $(\Omega^u(X, Y)\psi)(x)$ is determined by $\Omega^u(X, Y)_x$ and $\psi(x)$ alone. Since Ω^u is linear, it must be an endomorphism in the fibers.

This allows us both to compute the renormalized first Chern form $R_1^Q = \mathrm{str}^Q(\Omega^u)$ and to relate $\mathrm{d}(\mathrm{str}^Q(\Omega^u))$ and $\mathrm{str}^Q([\tilde{\nabla}^u, \Omega^u]) = 0$. Here $Q = L^2$, with L a first-order differential operator as in (3.3). Since Q (which in [4] is the square of the Dirac operator in the fibers) is a second-order differential operator, we have the asymptotic expansion for the heat kernel $\mathrm{e}^Q(\varepsilon, x, x)$:

$$\mathrm{e}^Q(\varepsilon, x, x) = \frac{1}{(4\pi)^{(\dim M)/2}} \left(\sum_{j=-\dim M}^{J} \alpha_j(x)\epsilon^{j/2} + \mathrm{O}(\varepsilon^{(J+1)/2}) \right).$$

$\alpha_j(x) \in \mathrm{Hom}(E_x, E_x)$ is locally computable from the metric on M_b at x and the symbol of Q at x. Ω^u is an endomorphism and Q is a differential operator, so $\mathrm{str}(\Omega^u \mathrm{e}^{\varepsilon Q})$ has an asymptotic expansion in ε with no logarithmic terms, and $\mathrm{str}^Q(\Omega^u) = \mathrm{str}^{Q,\mu}(\Omega^u)$ is consequently independent of μ. In fact, by the standard "remarkable cancellations" of local index theory, $\mathrm{str}(\Omega^u \mathrm{e}^{-\varepsilon Q})$ is O(1) as $\varepsilon \to 0$. Thus we can replace $\mathrm{Lim}^{\mu}_{\varepsilon\to 0}$ in the definition of the renormalized first Chern form $R_1^Q = R_1^{Q,\mu}$ by an ordinary limit:

$$\begin{aligned} R_1^Q &= \mathrm{str}^Q(\Omega^u) = \lim_{\varepsilon\to 0} \mathrm{str}(\Omega^u \mathrm{e}^{-\varepsilon Q}) = \lim_{\varepsilon\to 0} \int_M \mathrm{str}(\Omega_x^u \mathrm{e}^Q(\varepsilon, x, x)) \\ &= \frac{1}{(4\pi)^{(\dim M)/2}} \int_M \mathrm{str}^{\mu}(\Omega_x^u \alpha_0(x)), \end{aligned} \tag{7.12}$$

where we have used that Ω^u is a homomorphism in the fibers. Also,

$$\begin{aligned} &(4\pi)^{(\dim M)/2}\, \mathrm{d}\, \mathrm{str}^Q(\Omega^u) \\ &\quad = \mathrm{d} \int_M \mathrm{str}(\Omega_x^u \alpha_0(x)) = \int_M \mathrm{str}([\tilde{\nabla}, \Omega_x^u \alpha_0(x)]) \\ &\quad = \int_M \mathrm{str}([\tilde{\nabla}, \Omega_x^u])\alpha_0 + \Omega_x^u[\tilde{\nabla}_x, \alpha_0(x)]) = \int_M \mathrm{str}(\Omega_x^u[\tilde{\nabla}_x, \alpha_0]). \end{aligned}$$

Thus the obstruction to $R_1^Q = \mathrm{str}^Q(\Omega^u)$ being closed is given by the integral of $\mathrm{str}(\Omega^u[\tilde{\nabla}_x, a_0])$. The integrand is a local expression except in its dependence on $\tilde{\nabla}_x$.

Corollary 6. *In the Bismut–Freed setting of a fibration $\pi : Z \to B$ as above, assume that the metric on $\pi^{-1}(b)$ is flat and the connection on E is flat in fiber directions in $\pi^{-1}(b)$. Then the Bismut–Freed curvature at b reduces to a Wodzicki residue; i.e. in the notation of* Theorem 3,

$$\Omega^{\mathrm{Det},\mu}(M,N) = R^{Q,\nabla^{\varepsilon}}(M,N)$$

Proof. Since $\alpha_0 \equiv 0$ under the hypotheses, it follows from (7.12), that the term $R_1^{Q,\mu}$ in Theorem 3 vanishes and hence only the residue terms remain. □

The significance of the Corollary is that the curvature of the Bismut–Freed connection on the determinant line bundle of a family of Dirac operators, given by

$$\Omega^{\mathrm{Det},0} = \left[\int_M \hat{A}(\Omega^F)\,\mathrm{ch}(\Omega^W)\right]_{[2]}, \tag{7.13}$$

does not have this vanishing property; here Ω^F is the curvature of ∇^F. Thus Theorems 3 and 5 split the Bismut–Freed curvature into two terms. The first term $-R_1^{Q,\mu} = -\mathrm{str}^Q(\Omega^u)$, the analogue of the finite dimensional curvature, is localized on the fiber in the sense of the Corollary. The second obstruction term is a Wodzicki residue, which by Wodzicki's work is locally computed from the symbol of the (non-local) PDOs in Theorem 3. Thus the Bismut–Freed curvature breaks into two terms with locality properties in these technical senses.

Remark.

(1) We can define a Chern character form as $\sum_k \mathrm{str}^Q(\Omega^k)/k!$ for weighted bundles, and hence Chern forms via Newton's formulas. These forms will not be closed in general, and their significance is unclear.
(2) Theorems 3 and 5 compute the infinite dimensional obstruction to the finite dimensional equality of the curvature on the determinant bundle with (minus) the first Chern form on the original vector bundle. The different looking obstructions in these theorems are related by the fact that renormalized limits of expressions of the type $\langle A_0 A_1, \ldots, A_k\rangle_{k,\varepsilon,Q_0}$ can be expressed in terms of Wodzicki residues. More precisely, in Appendix B we show that the coefficients of divergent terms in the asymptotics of $\langle I, [\nabla^{\mathcal{E}}, L], [\nabla^{\mathcal{E}}, L]\rangle_{\varepsilon,2,Q_0}$ as $\varepsilon \to 0$ are combinations of Wodzicki residues.
(3) In fact, Proposition 4 is a more refined result than Theorems 3 and 5. Indeed, zeta function regularization at zero only detects logarithmic divergences, while heat kernel regularization keeps track of all divergences in ε in fractional powers of ε.

8. The Bismut–Freed connection and the curvature of the determinant bundle over the manifold of almost complex structures

In this section, we apply the theory of Section 7 to study the Bismut–Freed connection on the fibration associated to the string theory example of diffeomorphisms acting on the space of almost complex structures $\mathcal{A}(\Lambda)$ on a surface. We show that this connection agrees with a classical connection in Teichmüller theory, and we compute the renormalized first Chern form for the infinite dimensional bundle.

Let Λ be a smooth closed Riemannian surface of genus greater than one, and fix a Sobolev index $s > 1$. As in Section 2, Example (iii), we set

$$\mathcal{E}^+ = \mathcal{A}(\Lambda) \times H^{s+1}(T\Lambda),$$
$$\mathcal{E}^- = T\mathcal{A}^s|_{\mathcal{A}(\Lambda)} = \bigcup_{J\in\mathcal{A}(\Lambda)} \{H \in H^s(T_1^1\Lambda), JH = -HJ\}.$$

8.1. *Almost complex structures on the bundles $\mathcal{E}^\pm$*

Each of the real bundles $\mathcal{E}^\pm$ over $\mathcal{A}(\Lambda)$ has an almost complex structure. On $\mathcal{E}^+$, the almost complex structure is defined on the fiber above $J \in \mathcal{A}(\Lambda)$ by J itself:

$$\mathcal{J}_J^+(u) \equiv Ju, \quad u \in H^{s+1}(T\Lambda).$$

Similarly, the action

$$\mathcal{J}_J^-(H) = H \cdot J, \quad H \in T_J\mathcal{A}^s(\Lambda)$$

is an almost complex structure on $\mathcal{E}^-$.

Let $\mathcal{M}_{-1}^s(\Lambda)$ be the space of H^s Riemmanian metrics on Λ with curvature -1, and set

$$\Phi : \mathcal{A}^s(\Lambda) \to \mathcal{M}_{-1}^s(\Lambda), \quad \Phi(J) = g_J, \tag{8.1}$$

where g_J is the unique Riemannian metric g_J on Λ with curvature -1 in the conformal class defined by J. Φ is a diffeomorphism between the Hilbert manifolds $\mathcal{A}^s(\Lambda)$ and $\mathcal{M}_{-1}^s(\Lambda)$, and the derivative of Φ at J in the direction N is given by $(D_J\Phi(N))_{ab} = (g_J)_{ac}(N_J)_b^c$ [20]. For α_J as in (3.1), the operator

$$P_g = P_{g_J} \equiv D_J\Phi \circ \alpha_J \circ D_{g_J}\Phi^{-1} \tag{8.2}$$

plays a fundamental role in the Faddeev–Popov procedure for string theories (see [1]).

Lemma 7. *The bundle map $\alpha : \mathcal{E}^+ \to \mathcal{E}^-$ defined in (3.1) is compatible with the almost complex structures $\mathcal{J}^\pm$ in the sense that*

$$\alpha_J(\mathrm{Ker}(\mathcal{J}_J^+ - \mathrm{i})) = \mathrm{Ker}(\mathcal{J}_J^- - \mathrm{i}),$$
$$\alpha_J(\mathrm{Ker}(\mathcal{J}_J^- + \mathrm{i})) = \mathrm{Ker}(\mathcal{J}_J^- + \mathrm{i}) \quad \forall J \in \mathcal{A}(\Lambda).$$

Moreover, α_J is a first-order elliptic operator.

Proof. We first show that α_J is first-order elliptic. In isothermal coordinates for g, the complexified operator $P_g^{\mathbf{C}}$ is [1]

$$P_g^{\mathbf{C}}\left(u^{\bar z}\frac{\partial}{\partial \bar z} + u^z\frac{\partial}{\partial z}\right) = \partial_z u^{\bar z}\frac{\partial}{\partial \bar z} \otimes \mathrm{d}z + \partial_{\bar z} u^z\frac{\partial}{\partial z} \otimes \mathrm{d}\bar z, \tag{8.3}$$

so $P_g^{\mathbf{C}} = \bar\partial \oplus \partial$ where $\bar\partial$ is the Cauchy–Riemann operator. P_{g_J} is therefore first-order elliptic, and hence so is α_J, since its principal symbol differs from P_{g_J}'s by the isomorphisms in (8.2).

It is easy to check that $\mathrm{Ker}(\mathcal{J}_J^+ - \mathrm{i}) = \{u^z\partial/\partial z\}$ and $\mathrm{Ker}(\mathcal{J}_J^+ + \mathrm{i}) = \{u^{\bar z}\partial/\partial \bar z\}$, and that

$$\mathrm{Ker}(\mathcal{J}_J^+ - \mathrm{i}) = \left\{H_{\bar z}^z \frac{\partial}{\partial z}\mathrm{d}\bar z\right\}, \qquad \mathrm{Ker}(\mathcal{J}_J^+ + \mathrm{i}) = \left\{H_z^{\bar z} \frac{\partial}{\partial \bar z}\mathrm{d}z\right\}.$$

Indeed, since $J_1^1 = J_2^2 = 0$ and $J_2^1 = -1$ in isothermal coordinates, we have

$$(\mathcal{J}_J^- H)_z^{\bar z} = (HJ)_z^{\bar z} = \tfrac12((HJ)_2^1 - \mathrm{i}(HJ)_1^1) = \tfrac12(H_1^1 J_2^1 - \mathrm{i}H_2^1 J_1^2) = \tfrac12(-H_1^1 - \mathrm{i}H_2^1)$$
$$= -\tfrac12\,\mathrm{i}(H_2^1 - \mathrm{i}H_1^2) = -\mathrm{i}H_z^{\bar z}.$$

Similarly, $(\mathcal{J}_J^+ H)_{\bar z}^{\bar z} = \mathrm{i}H_{\bar z}^{\bar z}$. The lemma then follows from (8.3), since

$$u \in \mathrm{Ker}(\mathcal{J}_J^+ - \mathrm{i}) \Rightarrow u^{\bar z} = 0 \Rightarrow (P_{g_J}(u))_z^{\bar z} = 0 \Rightarrow P_{g_J}(u) \in \mathrm{Ker}(\mathcal{J}_J^- - \mathrm{i}). \qquad \square$$

8.2. Hermitian metrics on $\mathcal{E}^\pm$

$\mathcal{E}^\pm$ have the L^2 Riemannian metrics $\gamma^\pm$ given be (3.2a) and (3.2b), which are compatible with the almost complex structures $\mathcal{J}^\pm$. Indeed, for tangent vector fields u, v on Λ, we have

$$\langle \mathcal{J}^+ u, \mathcal{J}^+ v\rangle_J^+ = \langle Ju, Jv\rangle_J^+ = \langle u, v\rangle_J^+,$$

since g_J is compatible with J. Similarly, for (1, 1) tensors H, K on Λ, we have

$$\langle \mathcal{J}^- H, \mathcal{J}^- K\rangle_J^- = \langle HJ, KJ\rangle_J^+ = \int_\Lambda \mathrm{d}\mu_J(x)\mathrm{tr}(HJJ^*K^*) = \int_\Lambda \mathrm{d}\mu_J(x)\mathrm{tr}(HK^*),$$

since $J^* = -J$ and $J^2 = -1$.

Using the family of elliptic operators $\{Q_J \equiv Q_J^+ \oplus Q_J^- \equiv \alpha_J^*\alpha_J \oplus \alpha_J^*\alpha_J,\ J \in \mathcal{A}(\Lambda)\}$, we have H^s metrics $\gamma^{s,\pm}$ defined on the fiber $\mathcal{E}_J^\pm$ above J by

$$\langle u, v\rangle_J^{s,\pm} \equiv \langle (Q_J^\pm + 1)^s u, v\rangle_J^\pm = \langle (Q_J^\pm + 1)^{s/2} u, (Q_J^\pm + 1)^{s/2} v\rangle_J^\pm. \tag{8.4}$$

8.3. Connections on $\mathcal{E}^\pm$

We now define L^2 and H^s connections on $\mathcal{E}^\pm$. As $\mathcal{E}^+$ is trivial, let $\nabla^+ \equiv \mathrm{d} + \theta^+$ where

$$\theta^+(N) \equiv \tfrac12 NJ, \tag{8.5a}$$

for $N \in T_J\mathcal{A}(\Lambda)$, $J \in \mathcal{A}(\Lambda)$. Here NJ acts on $u \in H^{s+1}(T\Lambda)$ by $NJu(x) \equiv N(x)\cdot J(x)(u(x))$, with "·" denoting matrix multiplication. Since $N, J \in C^\infty(T_1^1\Lambda)$, multiplication by NJ preserves $H^{s+1}(T\Lambda)$, so θ^+ is a $\mathrm{Hom}(H^{s+1}(T\Lambda), H^{s+1}(T\Lambda))$-valued one-form on $\mathcal{A}$.

The local charts on the manifolds $\mathcal{A}^s(\Lambda)$ and $\mathcal{A}(\Lambda)$, given pointwise by the matrix exponential map as in Section 2, induce a local trivialization of $\mathcal{E}^-$ over the base space

$\mathcal{A}(\Lambda)$ with fibers $T_J\mathcal{A}^s(\Lambda)$, $J \in \mathcal{A}(\Lambda)$. In a local chart at $J \in \mathcal{A}(\Lambda)$, we set $\nabla^- \equiv \mathrm{d}+\theta^-$, with

$$\theta^-(N) \equiv -\tfrac{1}{2}J\{N, \cdot\} \tag{8.5b}$$

for $N \in T_J\mathcal{A}(\Lambda)$. Here $\{M, N\} \equiv MN + NM$. θ^- is a $\mathrm{Hom}(H^s(T_1^1\Lambda), H^s(T_1^1\Lambda))$-valued one-form on $\mathcal{A}(\Lambda)$, since we again matrix multiply elements in $H^s(T_1^1\Lambda)$ by elements in $C^\infty(T_1^1\Lambda)$. This connection corresponds to the "algebraic connection" defined in [20, (5.6)].

Lemma 8.

(1) *$\nabla^\pm$ are compatible with the L^2-metrics $\gamma^\pm$ and with the almost complex structures $\mathcal{J}^\pm$ in horizontal directions. In other words, the L^2 superconnection ∇ is Kähler in horizontal directions.*

(2) *$\nabla^{s,\pm} \equiv (Q^\pm + I)^{-s/2}\nabla^\pm(Q^\pm + I)^{s/2}$ are compatible with the H^s-metrics $\gamma^{s,\pm}$ and with the almost complex structures $\mathcal{J}^\pm$ and $\mathcal{J}^-$ in horizontal directions. In particular, the connection*

$$\nabla^s \equiv (\nabla^{s,+})^* \otimes 1 + 1 \otimes \nabla^{s,-}$$

is Kähler in horizontal directions.

Remark. It is shown in [20, Theorem 5.2.2] that in horizontal directions, ∇^- equals the L^2-Levi-Civita connection on the manifold of almost complex structures.

Proof.

(1) The compatibility of ∇^- with $\mathcal{J}^-$, γ^- is shown in [20, Theorems 5.2.1 and 5.2.2]. We adapt this proof to ∇^+ and refer the reader to [20] for details.

To prove the compatibility of ∇^+ with γ^+, first note that the derivative of the map $g \mapsto \mu_g$ sending a Riemannian metric g on Λ to the corresponding volume from μ_g vanishes in the direction of a traceless covariant two tensor. Indeed, we have $D_h(\mu_g) = \frac{1}{2}\,\mathrm{tr}_g(h)\mu_g = 0$. For any horizontal vector field N at J, we set $n \equiv D_J\Phi(N)$, for Φ in (8.1). n is a traceless covariant two tensor [20, Theorem 2.5.6]. For $u, v \in \Gamma(\mathcal{E}^+)$, we have

$$\begin{aligned} N\langle u, v\rangle_J^+ &= n\int_\Lambda \mathrm{d}\mu_{g_J} g_{J_{ab}} u^a v^b \\ &= \int_\Lambda \mathrm{d}\mu_{g_J} n_{ab} u^a v^b + \int_\Lambda \mathrm{d}\mu_{g_J} g_{J_{ab}} D_n u^a v^b + \int_\Lambda \mathrm{d}\mu_{g_J} g_{J_{ab}} u^a D_n v^b \\ &= \int_\Lambda \mathrm{d}\mu_{g_J} g_{J_{ac}} N_d^c J_b^d u^a v^b + \langle D_N u, v\rangle_J^+ + \langle u, D_N v\rangle_J^+ \\ &= \langle NJu, v\rangle_J^+ + \langle D_N u, v\rangle_J^+ + \langle u, D_N v\rangle_J^+ \\ &= \frac{1}{2}\langle NJu, v\rangle_J^+ - \frac{1}{2}\langle u, JNv\rangle_J^+ + \langle D_N u, v\rangle_J^+ + \langle u, D_N v\rangle_J^+ \\ &= \langle \nabla_N^+ u, v\rangle_J^+ + \langle u, \nabla_N^+ v\rangle_J^+, \end{aligned} \tag{8.6}$$

where we have used $JN = -NJ$ and $N^* = N$.

For the compatibility with the almost complex structure $\mathcal{J}^+$, we have

$$[\nabla_N^+, \mathcal{J}^+]u = D_N(Ju) + \tfrac{1}{2}NJ^2u - JD_N u - \tfrac{1}{2}JNJ = Nu - \tfrac{1}{2}Nu + \tfrac{1}{2}J^2Nu = 0.$$

(2) This is a straightforward consequence of (1), once we check the compatibility of $Q_J^{\pm}$ with the almost complex structures $\mathcal{J}_J^{\pm}$, which follows from Lemma 7. □

8.4. A half-weighted vector bundle

The bundle

$$\mathrm{Hom}(\mathcal{E}^{+,1,0}, \mathcal{E}^{-,1,0}) \equiv (\mathcal{E}^{+,1,0})^* \otimes \mathcal{E}^{-,1,0}$$

now has a connection $\nabla^s \equiv (\nabla^{s,+})^* \otimes 1 + 1 \otimes \nabla^{s-}$, which is horizontally Kähler. In a local chart, we have $\nabla^s = \mathrm{d} + \theta^s = \mathrm{d} + \theta^{s,-} - \theta^{s,+}$, so we can equivalently view ∇^s as a superconnection on the superbundle

$$\mathcal{E}^{1,0} \equiv \mathcal{E}^{+,1,0} \otimes \mathcal{E}^{-,1,0}. \tag{8.7}$$

The family

$$J \to L_J^{1,0} \equiv \begin{bmatrix} 0 & L_J^- \equiv \partial_J \\ L_J^+ \equiv \bar{\partial}_J & 0 \end{bmatrix},$$

where $\bar{\partial}_J$ is the Cauchy–Riemann operator for (Λ, J), defines a section of the bundle $\mathrm{Ell}(\mathcal{E}^{1,0})$. By Lemma 8, $L_J^{1,0}$ is a self-adjoint elliptic operator for the Hermitian product built from the almost complex structure $\mathcal{J} \equiv \mathcal{J}^+ \oplus \mathcal{J}^-$ and the scalar product $\langle\cdot,\cdot\rangle_J^+ \oplus \langle\cdot,\cdot\rangle_J^-$ (cf. [1,20] for a string theory perspective). Thus $(\mathcal{E}^{1,0}, L^{1,0})$ is a half-weighted vector bundle. $Q^{\pm 1,0} \equiv L^{\mp}L^{\pm}$ are positive self-adjoint sections of $\mathrm{Ell}(\mathcal{E}^{\pm,1,0})$. Here $L^{\mp}$ is either the L^2 or the H^s adjoint of $L^{\pm}$ with respect to the inner products (8.4). This data determines a weighted superbundle $(\mathcal{E}^{1,0}, Q^{1,0} \equiv Q^{+,1,0} \oplus Q^{-,1,0})$.

8.5. The first Chern forms of $\mathcal{E}^{\pm,1,0}$

Lemma 9. *Let $(\mathcal{E}, Q)$ be a weighted vector bundle with an almost complex structure $\mathcal{J}$ compatible with Q (i.e. $Q\mathcal{J} = \mathcal{J}Q$), let $(\mathcal{E}^{1,0}, Q^{1,0})$ denote its $(1,0)$ part, and let $A \in \Gamma(CL(\mathcal{E}))$ satisfy $A\mathcal{J} = \mathcal{J}A$. Then*

$$\mathrm{tr}^{Q^{1,0}}(A^{1,0}) = \mathrm{tr}^Q(A) + \mathrm{i}\,\mathrm{tr}^Q(\mathcal{J}A).$$

Proof. Let $A^{1,0}$ be the $(1,0)$ part of $A^{\mathbf{C}}$, the fiberwise complexification of A with respect to $\mathcal{J}$. It is standard that $\mathrm{tr}(A^{1,0}) = \mathrm{tr}(A) + \mathrm{i}\,\mathrm{tr}(JA)$. Then

$$\begin{aligned}\mathrm{tr}^{Q^{1,0}}(A^{1,0}) &= \mathrm{Lim}_{z\to 0}\,\mathrm{tr}(A^{1,0}(Q^{1,0})^{-z}) = \mathrm{Lim}_{z\to 0}\,\mathrm{tr}((AQ^{-z})^{1,0}) \\ &= \mathrm{Lim}_{z\to 0}\mathrm{tr}(AQ^{-z}) + \mathrm{i}\,\mathrm{Lim}_{z\to 0}\,\mathrm{tr}(\mathcal{J}AQ^{-z}) \\ &= \mathrm{tr}^Q(A) + \mathrm{i}\,\mathrm{tr}^Q(\mathcal{J}A).\end{aligned}$$

□

We now compute the curvature of $\nabla^{s,\pm}$ on $\mathcal{E}^{\pm}$.

Lemma 10. *For $s > 1$, the curvatures $\Omega^{s,\pm}$, of the connections $\nabla^{s,\pm}$ are zero-order PDOs given by*

$$\Omega^{s,+}(M,N)H = -\tfrac{1}{4}(Q^+ + I)^{-s/2}[M,N]_{\mathrm{op}}H(Q^+ + I)^{s/2},$$
$$\Omega^{s,-}(M,N)H = (Q^- + I)^{-s/2}(-\tfrac{1}{2}[M,N]_{\mathrm{op}}H + \tfrac{1}{2}(-MHN + NHM) - \tfrac{1}{4}[\{M,H\},\{N,H\}])(Q^- + I)^{s/2},$$

for $M, N, H \in T_J\mathcal{A}(\Lambda)$.

Here $[M,N]_{\mathrm{op}}$ denotes the multiplication operator in the fiber over J associated to the bracket (pointwise over Λ) of the matrices M, N. In contrast, $[M,N]$ denotes the bracket of vector fields on $\mathcal{A}$ which are given by local extensions of the tangent vectors M, N at J. At a fixed J, we may extend M, N so that $[M,N] = 0$.

Proof. We prove the first equality only, since the second is similar. Using $M(J) = M$, $JM = -MJ$, $J^2 = -\mathrm{Id}$ and similar formulas for N, we have

$$\begin{aligned}\Omega^{s,+}(M,N) &= (\nabla^{s,+})^2(M,N) = [\nabla^{s,+}_M, \nabla^{s,+}_N] - \nabla^{s,+}_{[M,N]}\\ &= (Q^+ + I)^{-s/2}([\nabla^+_M, \nabla^+_N] - \nabla^+_{[M,N]})(Q^+ + I)^{s/2}\\ &= (Q^+ + I)^{-s/2}(\mathrm{d}\theta^+(M,N) + \theta^+ \wedge \theta^+(M,N))(Q^+ + I)^{s/2}\\ &= (Q^+ + I)^{-s/2}(M(\theta^+(N)) - N(\theta^+(M)) - \theta^+([M,N])\\ &\quad + \theta^+ \wedge \theta^+(M,N))(Q^+ + I)^{s/2}\\ &= (Q^+ + I)^{-s/2}(-\tfrac{1}{2}[M,N]_{\mathrm{op}} + \tfrac{1}{4}[MJ, NJ])(Q^+ + I)^{s/2}\\ &= (Q^+ + I)^{-s/2}(-\tfrac{1}{4}[M,N]_{\mathrm{op}})(Q^+ + I)^{s/2}. \qquad \square\end{aligned}$$

Proposition 11. *The weighted first Chern form R_1^Q on the weighted vector bundle $(\mathcal{E}^{1,0}, Q^{1,0})$ with the connections ∇^s is independent of the parameter μ used in the renormalization procedure and independent of $s > 1$. For $M, N \in T_J\mathcal{A}(\Lambda)$, we have*

$$R_1^Q(M,N) = \mathrm{i}\,\mathrm{tr}^{Q^-}(\tfrac{1}{2}J[M,N]_{\mathrm{op}} - \tfrac{1}{2}(-M(\cdot)N + N(\cdot)M) + \tfrac{1}{4}J[\{M,\cdot\},\{N,\cdot\}]) + \tfrac{1}{4}\,\mathrm{i}\,\mathrm{tr}^{Q^+}(J[M,N]_{\mathrm{op}}).$$

The traces are taken with respect to the L^2 inner products. Note that, in agreement with Corollary 6, the curvature on the associated determinant bundle is the Q-weighted trace of a multiplication operator.

Proof. By Lemma 7, Q^+ commutes with the almost complex structure, so the $(1,0)$ part $\Omega^{s,+,1,0}(M,N)$ of $\Omega^{s,+}(M,N)$ satisfies $\Omega^{s,+,1,0}(M,N) = (\Omega^{s,+}(M,N))^{1,0}$. Applying Lemmas 9 and 10, we find

$$\begin{aligned}\mathrm{trs}^{Q^{+^{1,0}},\mu}(\Omega^{s,+,1,0}(M,N)) &= \mathrm{trs}^{Q^+,\mu}(\Omega^{s,+}(M,N)) + \mathrm{i}\,\mathrm{trs}^{Q^+,\mu}(\mathcal{J}_J^+\Omega^{s,+}(M,N))\\ &= -\tfrac{1}{4}\,\mathrm{i}\,\mathrm{tr}^{Q^+,\mu}(J[M.N]_{\mathrm{op}}).\end{aligned}$$

Note that $\mathrm{trs}^{Q^+,\mu}(\Omega^{s,+}(M,N)) = 0$, since the curvature form is a skew-symmetric endomorphism for fixed M, N. The H^s trace trs defined via the inner product $\langle\cdot,\cdot\rangle_J^{s,+}$ in the first line equals the L^2 trace in the second line, because the powers $(Q^+)^{\pm(s/2)}$ cancel in the computation of trs. Similarly, we obtain

$$\begin{aligned}
&\mathrm{trs}^{Q^{-1,0},\mu}(\Omega^{s,-,1,0}(M,N)) \\
&\quad = \mathrm{trs}^{Q^-,\mu}(\Omega^{s,-}(M,N)) + \mathrm{i}\,\mathrm{trs}^{Q^-,\mu}(\mathcal{J}_J^-\Omega^{s,-}(M,N)) \\
&\quad = \mathrm{i}\,\mathrm{tr}^{Q^-,\mu}(\tfrac{1}{2}J[M,N]_{\mathrm{op}} - \tfrac{1}{2}(-M(\cdot)N + N(\cdot)M) + \tfrac{1}{4}(J\{M,\cdot\},\{N,\cdot\}])
\end{aligned}$$

For a differential operator A, there is no logarithmic divergence in the asymptotics of $\mathrm{tr}(A\,\mathrm{e}^{-\varepsilon Q})$ as $\varepsilon \to 0$. Since μ keeps track of the logarithmic divergence, the renormalized traces above are independent of μ.

The result now follows from Eq. (5.8). □

Remark. A matrix $H \in T_J\mathcal{A}^s(\Lambda)$ satisfies $HJ = -JH$ so two such matrices H, K satisfy $HKJ = JHK$. Writing

$$J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

in isothermal coordinates, we see that HK is of the form

$$\begin{bmatrix} \alpha & \beta \\ \beta & -\alpha \end{bmatrix},$$

as is any even product of matrices in $T_J\mathcal{A}^s(\Lambda)$. Hence $J[M,N]_{\mathrm{op}}$ is of the form

$$\begin{bmatrix} \gamma & \delta \\ -\delta & \gamma \end{bmatrix}.$$

In contrast to an incorrect claim in [17], $\mathrm{tr}^{Q^+,\mu}(J[M,N]_{\mathrm{op}})$ need not vanish.

8.6. ∇^+ and the Bismut–Freed connection

We now show that the connection ∇^+ of (8.5a) coincides with the Bismut–Freed connection associated to the string theory fibration $\Lambda \to (\mathcal{A}\times\Lambda)/\mathcal{D} \to \mathcal{A}/\mathcal{D}$, where $\mathcal{D} = \mathrm{Diff}_0^{s+1}$ is the (Sobolev $s+1$) isotopy group of Λ. It is equivalent to work with the trivial fibration $\Lambda \to \mathcal{A}\times\Lambda \to \mathcal{A}$, and to consider only directions perpendicular to the action of $\mathcal{D}$ on $\mathcal{A}\times\Lambda$ with respect to the natural metric.

$$\|(h,v)\|_x^2 = \|h\|_{g_J}^2 + \|v\|_{g_J}^2, \tag{8.8}$$

where $h \in T\mathcal{A}$, $v \in T\Lambda$ and the projection $\pi : \mathcal{A}\times\Lambda \to \mathcal{A}$ has $\pi(x) = J$. Note that the role of ∇^- of (8.5b) is implicit, since it is the Levi-Civita connection for the metric on $T\mathcal{A}$.

Let ∇^{LC} be the Levi-Civita connection on $\mathcal{A} \times \Lambda$ for the metric (8.8). By (7.10), for $\psi \in \Gamma(T\Lambda)$, we must show that

$$\nabla_h^+ \psi = P^{T\Lambda} \nabla_h^{\mathrm{LC}} \psi, \tag{8.9}$$

where $P^{T\Lambda}$ is the orthogonal projection of $T(\mathcal{A} \times \Lambda)$ to $T\Lambda$. By the six term formula for the Levi-Civita connection, we have

$$\begin{aligned} 2\langle P^{T\Lambda} \nabla_h^{\mathrm{LC}} \psi, v\rangle &= 2\langle \nabla_h^{\mathrm{LC}} \psi, v\rangle = h\langle \psi, v\rangle + \psi\langle h, v\rangle - v\langle h, \psi\rangle + \langle [h, \psi], v\rangle \\ &\quad + \langle [v, h], \psi\rangle - \langle [\psi, v], h\rangle. \end{aligned} \tag{8.10}$$

On the right-hand side of (8.10), we may extend h, v arbitrarily near x, so we choose h to be horizontal and v to be vertical near x. Then

$$\langle h, v\rangle = \langle h, \psi\rangle = 0 \tag{8.11}$$

in (8.10).

Let ϕ_t^v be the vertical flow of v. Then

$$\begin{aligned} \langle [v, h], \psi\rangle &= \left\langle \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \phi_{-t,*}^v h, \psi \right\rangle = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \langle \phi_{-t,*}^v h, \psi\rangle \\ &= \langle h, [-v, \psi]\rangle = \langle [\psi, v], h\rangle; \end{aligned} \tag{8.12}$$

since $\mathcal{D}$ preserves the volume measure $\mathrm{d}\mu_{g_J}$, we may move $\mathrm{d}/\mathrm{d}t$ past the inner product in the first line. Combining (8.10)–(8.12) gives

$$\begin{aligned} 2\langle P^{T\Lambda} \nabla_h^{\mathrm{LC}} \psi, v\rangle &= h\langle \psi, v\rangle + \langle [h, \psi], v\rangle = \langle \nabla_h^+ \psi, v\rangle + \langle \psi, \nabla^+ v\rangle + \langle [h, \psi], v\rangle \\ &= \langle h(\psi), v\rangle + \langle \psi, h(v)\rangle + \langle [h, \psi], v\rangle, \end{aligned}$$

where we have used the third line of (8.6) in the last line. Moreover, $[h, \psi] = h(\psi) - \psi(h) = h(\psi)$, since $h \in T\mathcal{A}$ may be lifted to be constant in vertical directions. So we finally obtain

$$\begin{aligned} 2\langle P^{T\Lambda} \nabla_h^{\mathrm{LC}} \psi, v\rangle &= 2\langle h(\psi), v\rangle + \langle hJ\psi, v\rangle + \langle \psi, h(v)\rangle = 2\langle h(\psi), v\rangle + \langle hJ\psi, v\rangle \\ &= 2\langle \nabla_h^+ \psi, v\rangle, \end{aligned}$$

since the extension of v may be taken to be constant in vertical directions, and so $h(v) = 0$.

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Appendix A. Superconnection formalism

This appendix summarizes the superconnection formalization used in Section 6. Useful references are [5,19].

A.1. Super vector bundle valued forms

A super vector bundle $\mathcal{E}$ over a manifold B is a $\mathbf{Z}_2$ graded vector bundle $\mathcal{E} = \mathcal{E}^+ \oplus \mathcal{E}^-$ over B. Let $\Omega(B, \mathcal{E})$ be the space of $\mathcal{E}$-valued differential forms on B. Since $\Omega(B, \mathcal{E}) \simeq \Omega(B) \otimes_{C^\infty(B)} \Gamma(\mathcal{E})$, where $\Omega(B)$ is the exterior algebra of forms on B, an element of $\Omega(B, \mathcal{E})$ can be written as $\omega \otimes \sigma$ with $\omega \in \Omega(B)$, $\sigma \in \Gamma(\mathcal{E})$. The $\mathbf{Z}$ grading on $\Omega(B)$ induces a $\mathbf{Z}_2$ grading on $\Omega(B) = \Omega^+(B) \oplus \Omega^-(B)$ into forms of even and odd degree, which, with the $\mathbf{Z}_2$ grading on $\mathcal{E}$, yields a $\mathbf{Z}_2$ grading on $\Omega(B, \mathcal{E}) = \Omega^+(B, \mathcal{E}) \oplus \Omega^-(B, \mathcal{E})$, where

$$\Omega^+(B, \mathcal{E}) \equiv \Omega^+(B, \mathcal{E}^+) \oplus \Omega^-(B, \mathcal{E}^-), \quad \Omega^-(B, \mathcal{E}) \equiv \Omega^+(B, \mathcal{E}^-) \oplus \Omega^-(B, \mathcal{E}^+).$$

A.2. From a connection to a one-parameter family of superconnections

A superconnection is an odd first-order differential operator $\nabla : \Omega^\pm(B, \mathcal{E}) \to \Omega^\mp(B, \mathcal{E})$ which satisfies the Leibniz rule in the $\mathbf{Z}_2$ graded sense:

$$\nabla(\omega \otimes \sigma) = \mathrm{d}\omega \otimes \sigma + (-1)^{|\omega|}\omega \otimes \nabla\sigma.$$

A connection ∇ on $\mathcal{E}$ which preserves the $\mathbf{Z}_2$ grading defines a map

$$\nabla : \Gamma(\mathcal{E}^\pm)(\subset \Omega^\pm(B, \mathcal{E})) \to \Gamma(T^*B \oplus \mathcal{E}^\pm) \subset \Omega^\mp(B, \mathcal{E}),$$

which extends uniquely to a superconnection on $\mathcal{E}$.

The $\mathbf{Z}_2$ grading on $\mathcal{E}$ induces a $\mathbf{Z}_2$ grading on the bundle $\mathrm{Hom}(\mathcal{E}, \mathcal{E}) = \mathrm{Hom}^+(\mathcal{E}) \oplus \mathrm{Hom}^-(\mathcal{E})$, where the even bundle maps, the sections of $\mathrm{Hom}^+(\mathcal{E})$, preserve the $\mathbf{Z}_2$ grading on $\mathcal{E}$, and the odd bundle maps, the sections of $\mathrm{Hom}^-(\mathcal{E})$, take $\mathcal{E}^\pm$ to $\mathcal{E}^\mp$. A section L of $\mathrm{Hom}^-(\mathcal{E})$ induces an odd map

$$L : \Omega^\pm(B, \mathcal{E}) \to \Omega^\mp(B, \mathcal{E}), \omega \otimes \sigma \mapsto (-1)^{|\omega|}\omega \otimes L\sigma.$$

∇ and L induce a one-parameter family of superconnections $\nabla_t^L \equiv \nabla + \sqrt{t}L$, $t > 0$, on $\mathcal{E}$.

A.3. From a superconnection on $\mathcal{E}$ to a superconnection on $\mathrm{Hom}(\mathcal{E}, \mathcal{E})$

A superconnection ∇ on $\mathcal{E}$ induces a connection on $\mathrm{Hom}(\mathcal{E}, \mathcal{E})$ defined by

$$[\nabla, A] \equiv \nabla A - (-1)^{|A|}A\nabla,$$

where $|A| = 0$ if A is even and $|A| = 1$ if A is odd. If $A = L \in \Gamma(\mathrm{Hom}^-(\mathcal{E}))$ and ∇ is a superconnection induced by a $\mathbf{Z}_2$ grading preserving connection on $\mathcal{E}$, then

$$\begin{aligned}[\nabla, L](\omega \in \sigma) &\equiv \nabla(L(\omega \otimes \sigma)) + L(\nabla(\omega \otimes \sigma)) \\ &= \nabla((-1)^{|\omega|}\omega \otimes L\sigma) + L(\mathrm{d}\omega \otimes \sigma + (-1)^{|\omega|}\omega \otimes \nabla\sigma) \\ &= (-1)^{|\omega|}\,\mathrm{d}\omega \otimes L\sigma + (-1)^{|\omega|+1}\,\mathrm{d}\omega \otimes L\sigma + (-1)^{2|\omega|}\omega \otimes \nabla L\sigma \\ &\quad + (-1)^{2|\omega|+1}\omega \otimes L\nabla\sigma = \omega \otimes [\nabla, L]\sigma.\end{aligned}$$

In the last line, the bracket is an ordinary bracket.

Appendix B. Trace forms and Wodzicki residues

In the appendix, we express the divergences in the asymptotics of the trace forms $\langle A_0, A_1, \ldots, A_k\rangle_{\varepsilon,k,Q}$ as $\varepsilon \to 0$ in terms of Wodzicki residues. Such a relation is suggested by Theorems 3 and 5, which compute the obstruction to the equality of the determinant curvature and the renormalized first Chern form alternately as such a divergent term and as a Wodzicki residue, respectively. Such trace forms have occurred in quantum algebras studied by Jaffe et al. [10] and in local index theory in non-commutative geometry treated by Connes and Moscovici [7].

B.1. Notation

For $j \in \mathbf{N}$ and $A, Q \in CL(M, E)$ such that the Q has scalar top order symbol, $[A]_Q^j \in CL(M, E)$ is the operator defined inductively by

$$[A]_Q^0 \equiv A, \qquad [A]_Q^{j+1} \equiv [Q, [A]_Q^j].$$

We will often drop the subscript Q, and use notation from the body of the paper. Notice that the operator $[A]_Q^j$ is of order $a + j(q-1)$ where $a = \mathrm{ord}(A)$, $q = \mathrm{ord}(Q)$, and that $[A]_{\varepsilon Q}^j = \varepsilon^j [A]_Q^j$ for any $\varepsilon > 0$, $j \in \mathbf{N}$.

Lemma B.1 ([14], Lemma 4.2). *If $p, \varepsilon, N > 0$ satisfy $((N-a)/q) - p - \varepsilon > 0$, then*

$$\mathrm{e}^{-tQ}A = \sum_{j=0}^{N-1} \frac{(-t)^j}{j!}[A]_Q^j\,\mathrm{e}^{-tQ} + R_N(A, Q, t),$$

where for any $c > 0$ such that $Q+c$ is invertible, there exists $C > 0$ such that $\|R_N(A, Q, t)\cdot (Q+c)^p\| \leq Ct^{((N-a)/q)-p-\varepsilon}$.

Lemma B.2. *Given $A_0, A_1, \ldots, A_k \in CL(M, E)$ and $j_k \leq N_k \in \mathbf{N}$, there exist $N_1, N_2, \ldots, N_{k-1} \in \mathbf{Z}$ such that for $j_i \leq N_i$ and $\alpha_i \in \{0, 1\}$, $i = 1, \ldots, k$ with at least one α_i*

unequal to one, the operator

$$
\begin{aligned}
&A_0(R_{N_1}(A_1, Q, \sigma_0))^{1-\alpha_1}([A_1]_Q^{j_1})^{\alpha_1}\\
&\quad \times \cdots \left(R_{N_i}\left(A_i, Q, \sigma_0 + \sum_{l=1}^{i} \alpha_l \sigma_l\right)\right)^{1-\alpha_i} ([A_i]_Q^{j_i})^{\alpha_i}\\
&\quad \times \cdots \left(R_{N_k}\left(A_k, Q, \sigma_0 + \sum_{i=1}^{k} \alpha_i \sigma_i\right)\right)^{1-\alpha_k} ([A_k]_Q^{j_k})^{\alpha_k}
\end{aligned}
\tag{B.1}
$$

is trace-class with trace bounded by

$$
C \cdot \prod_{j=1}^{k} \left(\sigma_0 + \sum_{i=1}^{j} \alpha_i \sigma_j\right)^{(1-\alpha_j)((N_j-a_j)/(q-p_j))}
$$

for some positive constant C.

Proof. We proceed by induction on k. For $k = 1$, there is an integer p_0 such that $A_0 Q^{-p_0}$ is trace-class. By Lemma B.1, we can choose N_1 such that $Q^{p_0} R_{N_1}(A_1, Q, \sigma_0)$ is bounded by

$$
C\sigma_0^{((N_1-a_1)/q)-p_0},
\tag{B.2}
$$

where $a_1 = \mathrm{ord}(A_1)$ and C is a positive constant. Then $A_0 R_{N_1}(A_1, Q, \sigma_0)$ is trace-class with trace bounded by an expression similar to (B.2).

We now assume the lemma through $k - 1$ for the induction step. C will denote a constant which may change from line to line. By Lemma B.1, there exists $N_k \in \mathbf{Z}$ such that $R_{N_k}(A_k, Q, \sigma_0 + \sum_{i=1}^{k} \alpha_i \sigma_i)$ is bounded by $C(\sigma_0 + \sum_{i=1}^{k} \alpha_i \sigma_i)^{(N_k-a_k)/q}$. For $\alpha_k = 0$, by induction we can choose $N_1, \ldots, N_{k-1}$ such that

$$
A_0(R_{N_1}(A_1, Q, \sigma_0))^{1-\alpha_1}([A_1]_Q^{j_1})^{\alpha_1} \cdots R_{N_{k-1}}\left(A_{k-1}, Q, \sigma_0 + \sum_{l=1}^{k-1} \alpha_1 \sigma_l\right) [A_{k-1}]_Q^{j_{k-1}}
$$

is trace-class with trace bounded by

$$
C \cdot \prod_{j=1}^{k-1} \left(\sigma_0 + \sum_{i=1}^{j} \alpha_i \sigma_i\right)^{(1-\alpha_j)((N_j-a_j)/(q-p_j))}.
$$

It follows from Lemma B.1 that $R_{N_k}(A_k, A, \sigma_0 + \sum_{i-1}^{k} \alpha_i \sigma_i)$ is bounded in norm by $C \cdot (\sigma_0 + \sum_{i=1}^{k} \alpha_i \sigma_i)^{(N_k-a_k)/q}$. Hence (B.1) is bounded by

$$
C \cdot \prod_{j=1}^{k-1} \left(\sigma_0 + \sum_{i=1}^{j} \alpha_i \sigma_i\right)^{(1-\alpha_j)((N_j-a_j)/(q-p_j))} \cdot \left(\sigma_0 + \sum_{i=1}^{k} \alpha_i \sigma_i\right)^{(N_k-a_k)/q}.
$$

Now assume $\alpha_k = 1$. We can choose p_{k-1}, N_{k-1} large enough that $Q^{-p_{k-1}}[A_k]_Q^{j_k}$ is bounded and $((N_{k-1}-a_{k-1})/q)-p_{k-1}>0$. Then Lemma B.1 implies

$$\left\| R_{N_{k-1}}\left(A_{k_1}, Q, \left(\sigma_0+\sum_{l=1}^{k-1}\alpha_l\sigma_l\right) Q^{p_{k-1}}\right)\right\| \le C\left(\sigma_0+\sum_{l=1}^{k-1}\alpha_l\sigma_l\right)^{((N_{k-1}-a_{k-1})/q)-p_{k-1}}.$$

If $\alpha_{k-1}=0$, this estimate and the lemma for $k-2$ produces the upper bound

$$C\cdot\prod_{j=1}^{k-2}\left(\sigma_0+\sum_{i=1}^{j}\alpha_i\sigma_i\right)^{(1-\alpha_j)((N_j-a_j)/(q-p_j))}\cdot\left(\sigma_0+\sum_{i=1}^{k-1}\alpha_i\sigma_i\right)^{((N_{k-1}-a_{k-1})/q)-p_{k-1}}$$

for the trace. If $\alpha_{k-1}=1$, there exist p_{k-2}, N_{k-2} such that $Q^{-p_{k-2}}[A_{k-1}]_Q^{j_{k-1}}[A_k]_Q^{j_k}$ is bounded and $((N_{k-2}-a_{k-2})/q)-p_{k-2}>0$. Applying the above procedure gives the desired estimates. □

Proposition B.3. *Let* $A_0, A_1, \ldots, A_k \in CL(M,E)$. *There exist* $N_1, N_2, \ldots, N_k \in \mathbf{Z}$ *such that for* $\varepsilon>0$.

$$\begin{aligned}&\langle A_0, A_1, \ldots, A_k\rangle_{\sqrt{\varepsilon},k,Q}\\ &\quad=\sum_{j_1=0}^{N_1-1}\cdots\sum_{j_n=0}^{N_k-1}\frac{\varepsilon^{j_1+\cdots+j_n}}{j_1!\cdots j_k!}\int_0^1\cdots\int_0^1(-1)^{j_1+\cdots j_k}(\sigma_0)^{j_1}(\sigma_0+\sigma_1)^{j_2}\\ &\qquad\cdots(\sigma_0+\sigma_1+\cdots+\sigma_{k-1})^{j_k}\,\mathrm{tr}(A_0[A_1]_Q^{j_1}\cdots[A_k]_Q^{j_k}\,\mathrm{e}^{-\varepsilon Q})\,\mathrm{d}\sigma_0\cdots\mathrm{d}\sigma_k+\mathrm{o}(\varepsilon).\end{aligned}$$

Proof. Iterating Lemma B.1, we find

$$\begin{aligned}&\langle A_0, A_1, \ldots, A_k\rangle_{\varepsilon,k,Q}\\ &\quad=\sum_{j_1=0}^{N_1-1}\cdots\sum_{j_n=0}^{N_k-1}(-1)^{j_1+\cdots j_k}\frac{\varepsilon^{j_1+\cdots+j_k}}{j_1!\cdots j_k!}\int_0^1\cdots\int_0^1(\sigma_0)^{j_1}(\sigma_0+\sigma_1)^{j_2}\\ &\qquad\cdots(\sigma_0+\sigma_1+\cdots+\sigma_{k-1})^{j_k}\,\mathrm{tr}(A_0[A_1]_Q^{j_1}\cdots[A_k]_Q^{j_k}\,\mathrm{e}^{-\varepsilon Q})\,\mathrm{d}\sigma_0\\ &\qquad\cdots\mathrm{d}\sigma_k+R_{N_1,N_2,\ldots,N_k}(\varepsilon),\end{aligned}$$

with $R_{N_1,N_2,\dots,N_k}(\varepsilon)$ a finite linear combination of terms of the type

$$\int_{\sigma_0+\cdots+\sigma_k=1,\sigma_i\geq 0} \operatorname{tr}\left(A_0(R_{N_1}(A_1,Q,\sigma_0))^{1-\alpha_1}\left(\sum_{j_1=0}^{N_1}\frac{(-\sigma_0)^{j_1}}{j_1!}[A_1]_Q^{j_1}\right)^{\alpha_1}\right.$$
$$\times(R_{N_2}(A_2,Q,\sigma_2))^{1-\alpha_2}\left(\sum_{j_2=0}^{N_2}\frac{(-(\sigma_0+\alpha_1\sigma_1))^{j_2}}{j_2!}[A_2]_Q^{j_2}\right)^{\alpha_2}$$
$$\times\cdots(R_{N_k}(A_n,Q,\sigma_k))^{1-\alpha_k}\left(\sum_{j_n=0}^{N_n}\frac{(-(\sigma_0+\alpha_1\sigma_1+\cdots+\alpha_k\sigma_k))^{j_k}[A_k]_Q^{j_k}}{j_k!}\right)^{\alpha_k}$$
$$\left.\times\, \mathrm{e}^{-\varepsilon(\sigma_0+(1-\alpha_1)\sigma_1+\cdots+(1-\alpha_k)\sigma_k)Q}\right)\mathrm{d}\sigma_0\cdots\mathrm{d}\sigma_k,$$

with α_i equal to 0 or 1, and $(\alpha_1,\dots,\alpha_n)\neq(1,\dots,1)$. By Lemma B.2, $N_1, N_2, \dots, N_k$ can be chosen so that the integrals in $R_{N_1,N_2,\dots,N_k}(\varepsilon)$ converge and $R_{N_1,N_2,\dots,N_k}(\varepsilon)=o(\varepsilon)$. □

B.2. The asymptotics of regularized trace forms

We now investigate the asymptotic behavior or the trace forms as $\varepsilon\to 0$.

Theorem B.4. *Let $A_0,\dots,A_n\in CL(M,E)$. Then*

(i) *$\langle A_0,A_1,\dots,A_n\rangle_{\varepsilon,n,Q}$ has the following asymptotic expansion as $\varepsilon\to 0$:*

$$\langle A_0,A_1,\dots,A_n\rangle_{\varepsilon,n,Q}\sim\sum_{j=0}^{\infty}\alpha_j(A_0,A_1,\dots,A_n)\varepsilon^{\lambda_j}+\sum_{k=0}^{\infty}\beta_k(A_0,A_1,\dots,A_n)\varepsilon^k\log\varepsilon+\sum_{k=0}^{\infty}\lambda_k(A_0,A_1,\dots,A_n)\varepsilon^k,$$

where $\lambda_j=(j-a-\dim M)/q$ with $a\equiv\operatorname{ord}(A_0)+\cdots+\operatorname{ord}(A_n)$, $q\equiv\operatorname{ord}(Q)$, and $\alpha_j(A_0,A_1,\dots,A_n)$, $\beta_k(A_0,A_1,\dots,A_n)$, $\gamma_k(A_0,A_1,\dots,A_n)\in\mathbf{C}$.

(ii) *For $j\in\mathbf{N}$ with $\operatorname{Re}(\lambda_j)=(j-a-\dim M)/q<0$, there is a multi-index $(N_1,\dots,N_n)\in\mathbf{N}^n$ such that*

$$\alpha_j(A_0,A_1,\dots,A_n)=\sum_{j_1=0}^{N_1}\cdots\sum_{j_n=0}^{N_n}(-1)^{j_1+\cdots j_n}\left[\int_0^1\cdots\int_0^1\left(\sigma_1^{j_1}\cdots\sigma_n^{j_n}\times\frac{\Gamma((-j+a+\dim M)/q)+(j_1+\cdots+j_n)}{q\cdot j_1!j_2!\cdots j_n!}\right)\mathrm{d}\sigma_1\cdots\mathrm{d}\sigma_n\right]$$
$$\times\operatorname{res}(A_0[A_1]_Q^{j_1}\cdots[A_n]_Q^{j_n}Q^{((j-a-\dim M)/q)-(j_1+\cdots+j_n)}),$$

where res *denotes the Wodzicki residue.*

Remark. Assuming $\mathrm{Re}(\lambda_j) < 0$ ensures that $\Gamma(((-j+a+\dim M)/q)+(j_1+\cdots+j_n))$ is well defined for all $(j_1,\ldots,j_n) \in \mathbf{N}^n$.

The proof of the theorem depends on a lemma whose proof we include for completeness.

Lemma B.5. *Let $A \in CL(M,E)$ and $Q \in \mathrm{Ell}^+_{\mathrm{ord}>0}(M,E)$. There is the asymptotic expansion as $\varepsilon \to 0$*

$$\mathrm{tr}(A\,\mathrm{e}^{-\varepsilon Q}) \sim \sum_{j=0}^{\infty} \alpha_j(A)\varepsilon^{\lambda_j} + \sum_{k=0}^{\infty} \beta_k(A)\varepsilon^k \log\varepsilon + \sum_{k=0}^{\infty} \gamma_k(A)\varepsilon^k, \tag{B.3}$$

with $\alpha_j(A), \beta_k(A), \gamma_k(A) \in \mathbf{C}$, $a = \mathrm{ord}(A)$, $q = \mathrm{ord}(Q)$, and $\lambda_j = (j-a-\dim M)/q$. For j with $\mathrm{Re}(\lambda_j) < 0$ (e.g. $\mathrm{ord}(A) \notin \mathbf{Z}$), and for $k \in \mathbf{N}$, we have

$$\alpha_j(A) = \frac{\Gamma((-j+a+\dim M)/q)}{q}\,\mathrm{res}(AQ^{(-j-a-\dim M)/q}),$$

$$\beta_k(A) = (-1)^k \frac{q\,\mathrm{res}(AQ^{-k})}{(k-1)!}.$$

Proof. For $s \in \mathbf{C} - \{0,-1,-2,\ldots\}$, we have

$$\begin{aligned}
&q^{-1}\,\mathrm{res}(AQ^{-s})\\
&\quad = \mathrm{res}_{z=0}\,\mathrm{tr}(AQ^{-(z+s)}) = \mathrm{res}_{z=0}\left(\frac{1}{\Gamma(s+z)}\int_0^\infty t^{s+z-1}\,\mathrm{tr}(A\,\mathrm{e}^{-tQ})\,\mathrm{d}t\right)\\
&\quad = \mathrm{res}_{z=0}\left(\frac{1}{\Gamma(s)}\int_0^1 t^{s+z-1}\,\mathrm{tr}(A\,\mathrm{e}^{-tQ})\,\mathrm{d}t + \frac{1}{\Gamma(s)}\int_1^\infty t^{s+z-1}\,\mathrm{tr}(A\,\mathrm{e}^{-tQ})\,\mathrm{d}t\right)\\
&\quad = \sum_j \frac{\alpha_j(A)}{\Gamma(s)}\,\mathrm{res}_{z=0}\left(\int_0^1 t^{z+\lambda_j+s-1}\,\mathrm{d}t\right)\\
&\qquad + \sum_{k=0}^{\infty} \frac{\beta_k(A)}{\Gamma(s)}\,\mathrm{res}_{z=0}\left(\int_0^1 t^{k+z+s-1}\log t\,\mathrm{d}t\right)\\
&\quad = \Gamma(s)^{-1}\sum_j \alpha_j(A)\,\mathrm{res}_{z=0}\left[\frac{t^{z+\lambda_j+s}}{z+\lambda_j+s}\right]_0^1 = \Gamma(s)^{-1}\alpha_{-q\cdot s+a+\dim M}(A)
\end{aligned} \tag{B.4}$$

since $\lambda_j = -s$ iff $j = -qs+a+\dim M$. Notice that $s \in -\mathbf{N}$ iff $\lambda_j = (j-a-\dim M)/q \in \mathbf{N}$, which does not occur if $\mathrm{Re}(\lambda_j) < 0$. In this computation, we use the fact that the terms in (B.3) containing logarithmic divergences in ε or having integral powers of ε do not contribute to the residue at $z = 0$. Similarly, the $\int_1^\infty$ term in (B.4) does not contribute to the residue.

For $s=-l$, $l\in\mathbf{N}$, using $\Gamma(z)=z(z-1)\cdots(z-k+1)\Gamma(z-k)$ and $\Gamma(z)^{-1}\sim z$ as $z\to 0$, we find

$$\begin{aligned} q^{-1}\operatorname{res}(AQ^{-l}) &= \operatorname{res}_{z=0}\left(\frac{1}{\Gamma(z-l)}\int_0^\infty t^{z-(l+1)}\operatorname{tr}(A\,\mathrm{e}^{-tQ})\,\mathrm{d}t\right)\\ &= \operatorname{res}_{z=0}\left(\frac{z(z-1)\cdots(z-l+1)}{\Gamma(z)}\int_0^\infty t^{z-(l+1)}\operatorname{tr}(A\,\mathrm{e}^{-tQ})\,\mathrm{d}t\right)\\ &= -\sum_{k=0}^{\infty}\beta_k\operatorname{res}_{z=0}\left(z^2(z-1)\cdots(z-l+1)\frac{t^{z-l+k}}{(z-l+k)^2}\right)\\ &= (-1)^l(l-1)!\beta_l. \end{aligned}$$

□

Proof of the Theorem.

(i) The operator $A_0[A_1]_Q^{j_1}\cdots[A_n]_Q^{j_n}$ is a PDO of order at most $a+(j_1+\cdots+j_n)q$, so $\operatorname{tr}[A_0[A_1]_Q^{j_1}\cdots[A_n]_Q^{j_n}\,\mathrm{e}^{-\varepsilon Q})$ has an asymptotic expansion as in (B.3) with $\lambda_j=[(j-a-\dim M)/q]-(j_1+\cdots+j_n)$. By Proposition B.1, $\langle A_0,A_1,\ldots,A_n\rangle_{\varepsilon,n,Q}$ has an asymptotic expansion as in (B.3) with $\lambda_j=(j-a-\dim M)/q$. Let $\tilde\alpha_j(A_0,A_1,\ldots,A_n)$ be the coefficient of ε^{λ_j} in the asymptotic expansion of $\operatorname{tr}(A_0[A_1]_Q^{j_1}\cdots[A_n]_Q^{j_n}\,\mathrm{e}^{-\varepsilon Q})$ with $\lambda_j=[(j-a-\dim M)/q]-(j_1+\cdots+j_n)$.

(ii) By Lemma B.3, if $\operatorname{Re}(\lambda_j)-(j_1+\cdots+j_n)<0$ (e.g. if $\operatorname{Re}(\lambda_j)<0$), then

$$\begin{aligned} &\tilde\alpha_j(A_0,A_1,\ldots,A_n)\\ &\quad=\frac{\Gamma(-\lambda_j+(j_1+\cdots+j_n))}{q}\\ &\qquad\times\operatorname{res}(A_0[A_1]_Q^{j_1}\cdots[A_n]_Q^{j_n}Q^{((j-a-\dim M)/q)-(j_1+\cdots+j_n)}). \end{aligned}$$

Part (ii) of the theorem follows. □

References

[1] S. Albeverio, J. Jost, S. Paycha, S. Scarlatti, A Mathematical Introduction to String Theory, LMS Lecture Note Series, Vol. 225, Cambridge University Press, Cambridge, 1997.
[2] M. Arnaudon, S. Paycha, Regularisable and minimal orbits for group actions in infinite dimensions, Commun. Math. Phys. 191 (1998) 641–662.
[3] J.-M. Bismut, Localization formulae, super connections and the index theorem for families, Commun. Math. Phys. 103 (1986) 127–166.
[4] J.-M. Bismut, D. Freed, The analysis of elliptic families I, Commun. Math. Phys. 106 (1986) 159–176.
[5] N. Berline, E. Getzler, M. Vergne, Heat Kernels and Dirac Operators, Springer, Berlin, 1996.
[6] A. Cardona, C. Ducourtioux, J.-P. Magnot, S. Paycha, Weighted traces on algebras of pseudodifferential operators and geometry of loop groups, Inf. Dim. Anal. Quantum Prob. Rel. Fields, in press.
[7] A. Connes, H. Moscovici, The local index formula in non-commutative geometry, GAFA 5 (1995) 174–243.
[8] D. Freed, The geometry of loop groups, J. Diff. Geom. 28 (1988) 223–276.
[9] P.B. Gilkey, Invariance Theory, the Heat Equation, and the Atiyah–Singer Index Theorem, CRC Press, Boca Raton, 1995.

[10] A. Jaffe, A. Lesniewski, K. Osterwalder, Quantum K-theory. I. The Chern character, Commun. Math. Phys. 118 (1988) 1–14.

[11] C. Kassel, Le résidu non commutatif [d'après Wodzicki], Séminaire Bourbaki, Vol. 708, Astérisque, 1989, pp. 199–229.

[12] M. Kontsevich, S. Vishik, Determinants of elliptic pseudodifferential operators, Geometry and Functional Analysis, Max Planck Preprint, 1994.

[13] M. Kontsevich, S. Vishik, Geometry of determinants of elliptic operators, in: S. Gindikin, J. Lepowski, R.L. Wilson (Eds.), Functional Analysis on the Eve of the 21st Century, Vol. I, Progress in Mathematics, Birkhäuser, Basel, 1994, pp. 173–197.

[14] M. Lesch, On the non-commutative residue for pseudo-differential operators with log-polyhomogeneous symbols, Ann. Global Anal. Geom. 17 (1998) 151–187.

[15] Y. Maeda, S. Rosenberg, P. Tondeur, The mean curvature of gauge orbits, in: K. Uhlenbeck (Ed.), Global Analysis and Modern Mathematics, Publish or Perish, Houston, 1994, pp. 171–220; ;
Y. Maeda, S. Rosenberg, P. Tondeur, Minimal orbits of metrics, J. Geom. Phys. 23 (1997) 314–349.

[16] S. Paycha, Regularized pseudo-traces as a looking glass into infinite dimensional geometry, Inf. Dim. Anal., Quantum Prob., Rel. Fields 4 (2) (2001) 221–266.

[17] S. Paycha, S. Rosenberg, About infinite dimensional group actions and determinant bundles, in: H. Heyer, J. Marison (Eds.), Analysis on Infinite-dimensional Lie Groups and Algebras, World Scientific, Singapore, 1998, pp. 355–367.

[18] D. Quillen, Determinants of Cauchy–Riemmann operators over a Riemann surface, Funktsional Anal. i Prilozhen. 19 (1985) 37–41.

[19] D. Quillen, Superconnections and the Chern character, Topology 24 (1985) 89–95.

[20] T. Tromba, Teichmüller Theory in Riemannian Geometry, Birkhäuser, Boston, 1992.

[21] M. Wodzicki, Non-commutative Residue, Lecture Notes in Mathematics, Vol. 1289, Springer, Berlin, 1987.